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NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–7, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Greenville, Pa., May 13.

ILLINOIS, Galesburg, May 12–13.

INDIANA, Muncie, April 28–29.

IOWA, Ames, April 21–22.

KANSAS, Topeka, April 1.

KENTUCKY, Murray, April 28–29.

LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Ground, Md., May 13.

MICHIGAN, Ann Arbor, March 18.

MINNESOTA, Northfield, May 13.

MISSOURI, Springfield, April 28.

NEBRASKA, Lincoln, May 5.

NORTHERN CALIFORNIA, San Francisco, January 28.

OHIO, Columbus, April 8.

OKLAHOMA, Tulsa, February 10.

PHILADELPHIA, Bethlehem, Pa., December 2.

ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.

SOUTHEASTERN, Charleston, S.C., March 24–25.

SOUTHERN CALIFORNIA, Whittier, March 4.

SOUTHWESTERN, Alpine, Texas, May 2–3.

TEXAS, Abilene, March 31–April 1.

WISCONSIN, Milwaukee, May 6.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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MATHEMATICAL ASSOCIATION OF AMERICA

The following twenty-two persons have been elected to membership in the Association on applications duly certified:

C. K. ALEXANDER, Ph.D.(Calif. Inst. of Tech.) Asst. Prof., Acting Chm. of Dept., Occidental Coll., Los Angeles, Calif.

R. C. BLACKWELL, A.M.(North Carolina) Grad. student, Univ. of North Carolina, Chapel Hill, N.C.

A. O. BOATMAN, A.M. (Indiana), Prof., Carthage College, Carthage, Ill.

W. C. BOYRER, M.E., M.M.E.(Cornell) Accounting Engr., Retired, Consolidated Edison Co., Charleston, S.C.

W. B. BROWN, Ph.D.(Ohio State) Prof., Mississippi Woman's Coll., Hattiesburg, Miss.

A. H. DIAMOND, Ph.D.(California). Prof., Head of Dept., Oklahoma A. and M. Coll., Stillwater, Okla.

R. T. DONNELL, A.M.(Vanderbilt) Head of Dept., Cumberland Univ., Lebanon, Tenn.

Rev. W. G. DOYLE, M.S.(Catholic Univ.) Teacher, Bishop England High School, Charleston, S.C.

H. T. FLEDDERMANN, M.S.(Louisiana State) Asst. Prof., Loyola Univ., New Orleans, La.

R. A. GOODPASTURE, B.S. in C.E.(Colo. State Coll.) Jr. Engr., U.S. Bureau of Reclamation, Denver, Colo.

P. W. HALLETT, B.A.(Sydney Univ.) Deputy Headmaster, and Math. Master, Sydney Boys High School, Moore Park, Surry Hills, N.S.W., Australia

E. E. HEMENOVER, M.S.(Wyoming) Prof., Engineering, Pueblo Jr. Coll., Pueblo, Colo.

WILFRED KAPLAN, A.M.(Harvard) Teaching fellow, Rice Inst., Houston, Texas

J. L. KELLEY, A.M.(U.C.L.A.) Instr., Univ. of Virginia, Charlottesville, Va.

GASPERINE MILO, B.S.(New River State Coll.) Substitute Teacher, New River State College, Montgomery, W. Va.

ABBA V. Newton, Ph.D.(Chicago) Prof., Hartwick Coll., Oneonta, N.Y.

Rev. C. W. O'HARA, M.S. Prof., Math. and Physics, Heythrop Coll., Chipping Norton, Oxon, England

C. C. OURSLER, A.B.(Indiana) Prin., High School, Lancaster, Ill.

W. T. SCOTT, Ph.D.(Rice) Instr., Armour Inst. of Tech., Chicago, Ill.

C. L. SEEBECK, JR., A.M.(Harvard) Fellow, Univ. of North Carolina, Chapel Hill, N.C.

HARRY SILLER, M.S.(New York Univ.) Jr. statistical clerk, Social Security Board, Washington, D.C.

C. U. WETZIG, A.M. (Texas) Instr., A. and M. Coll., Magnolia, Ark.

M. A. ZORN, Dr. res mat.(Hamburg) Asso. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.

The Trustees have voted, 1, to hold the annual meeting of the Association in December 1940 at Louisiana State University, Baton Rouge, Louisiana, in conjunction with the annual meeting of the American Mathematical Society; 2, to appropriate the interest from the Chace Fund for a period of five years to the support of a proposed historical journal, a project which is in line with the interests of Chancellor A. B. Chace in his lifetime; 3, to guarantee $600 toward the expense of the International Congress of Mathematicians to be held at Cambridge, Massachusetts, in September 1940.

W. D. CAIRNS, *Secretary-Treasurer*

THE SIXTEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The sixteenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Louisiana State University, Baton Rouge, March 3–4, 1939. Sessions were held on Friday afternoon and Saturday morning. On Friday evening a joint dinner with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics was held. The chairman of the Section, Professor J. F. Thomson, presided at the Friday afternoon session and at the dinner. Vice-Chairman H. F. Schroeder presided at the Saturday morning session.

The attendance was about one hundred, including the following thirty-three members of the Association: A. A. Aucoin, W. G. Banks, Jr., E. T. Browne, H. E. Buchanan, D. S. Dearman, W. L. Duren, Jr., Virginia I. Felder, H. T. Fleddermann, Elizabeth Freas, F. C. Gentry, Charles Hopkins, H. T. Karnes, Dorothy McCoy, Janet McDonald, B. E. Mitchell, S. B. Murray, I. C. Nichols, Irene A. Nolan, Arthur Ollivier, R. L. O'Quinn, W. V. Parker, H. L. Quarles, F. A. Rickey, S. T. Sanders, S. T. Sanders Jr., H. F. Schroeder, P. C. Scott, C. D. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, Marelena White, M. C. Wicht.

The following officers were elected for the year 1939–40: Chairman, V. B. Temple, Louisiana College; Vice-Chairman for Louisiana, H. T. Fleddermann, Loyola University; Vice-Chairman for Mississippi, H. L. Quarles, University of Mississippi; Secretary, W. V. Parker, Louisiana State University. The meeting for 1940 is to be held at the University of Mississippi.

The Section was honored to have as guest speaker Professor E. T. Browne of the University of North Carolina. The two addresses which he gave contributed much to the value of the meeting. His first address on "Observations on the study and the teaching of mathematics" was given at the dinner Friday evening. At the Saturday morning session he spoke on "Quasi-k-commutative matrices."

The following are abstracts of Professor Browne's papers.

1. Professor Browne pointed out that in mathematics, probably even more than in other subjects, a capable teacher is necessary; and by a *capable* teacher is meant one who first of all has an adequate knowledge of his subject. Such knowledge is not gained merely by a course of study in school and college, but by constant reading on mathematics along with one's teaching. And further, if a teacher is to be successful he must see that his students really know how to study mathematics and that they learn how to make proper use of their memory. The speaker then mentioned three main criticisms that he had heard voiced concerning the study of mathematics: that mathematics is an exceptionally difficult subject except for the few who have a special talent for it; that mathematics is dead; and that it is of no use to the average man. He discussed these criticisms briefly, showing that none of them actually has any foundation in fact.

2. If A and B are two n-square matrices such that $AB-BA=0$, A and B are called *commutative*. If $AB-BA=C\neq 0$ where $CA-AC=0$, $CB-BC=0$, McCoy calls A and B *quasi-commutative*. This notion was generalized by Roth into what he calls mutually k-commutative matrices, which include the ordinary commutative matrices as mutually *one-commutative*, and McCoy's quasi-commutative matrices as mutually *two-commutative*. In this paper, Professor Browne suggested that a generalization be made in a somewhat different direction. Thus, let $AB-BA=C$ be *nilpotent*, and form the two sequences:

$$
\begin{array}{ll}
CA - AC = A_1, & CB - BC = B_1, \\
CA_1 - A_1C = A_2, & CB_1 - B_1C = B_2, \\
\cdots\cdots\cdots & \cdots\cdots\cdots \\
CA_{l-1} - A_{l-1}C = A_l, & CB_{k-1} - B_{k-1}C = B_k.
\end{array}
$$

Since C is nilpotent, both of these sequences necessarily terminate. If A_l and B_k, respectively, are the first matrices in these sequences which are zero, then if $k\geqq l$, we call A and B *quasi-k-commutative*. If $k=1$, it will be seen that A and B are quasi-commutative in the sense of McCoy, while if $C=0$, A and B are commutative. This paper is concerned with a study of pairs of quasi-k-commutative matrices A and B.

The following program was given:

1. "Squares escribed and inscribed to a triangle" by Professor B. E. Mitchell, Millsaps College.

2. "A proof of the addition theorems of trigonometry" by Professor W. L. Duren, Jr., Tulane University.

3. "The summation of finite series" by Albert Farnell, Louisiana State University, introduced by the Secretary.

4. "The least super sphere of a set" by Professor H. T. Fleddermann, Loyola University.

5. "Cremona involutions determined by a pencil of surfaces," by Professor F. C. Gentry, Louisiana Polytechnic Institute.

6. "Equations with coefficients in a modular field" by Professor Charles Hopkins, Tulane University.

7. "Tchebycheff approximations for decreasing functions" by Professor C. D. Smith, Mississippi State College.

8. "A problem in minimum values" by Professor F. A. Rickey, Louisiana State University.

9. "Equation of heat in the fins of air-cooled engines" by Professor W. B. Brown, Mississippi Woman's College, introduced by the Secretary.

10. "A formula on installment buying" by Professor P. K. Smith, Louisiana Polytechnic Institute.

11. "Some unit and zero identities" by Professor V. B. Temple, Louisiana College.

12. "The cevian tetrahedron and some of its remarkable points" by M. C. Wicht, Louisiana State University.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles.

1. Defining an escribed square as one having two vertices on one side of a given triangle and one each on the other sides, Professor Mitchell proceeds to investigate the escribed squares to a triangle. In general there are six, associated one pair with each of the three sides of the triangle. The right triangle has only four distinct escribed squares, the two pairs associated with the legs having coalesced. If an altitude is equal to its associated side one of the escribed squares for that side becomes infinite. The relations of the squares to the triangle and to each other are many and varied: homothetic, harmonic, and quasi-harmonic. When the squares are considered in triplets the relationship leads to Brocard geometry.

2. The textbooks in trigonometry usually prove the addition theorems of $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ first for the case where α and β are positive angles whose sum is acute. This is followed by induction to establish the formulas in general. In this paper Professor Duren gives a simple general proof without such an induction. The proof makes use of two coördinates systems and the general formula $A'B'=AB\cos\alpha$ for the projection of the directed line segment AB upon a directed line.

3. Mr. Farnell gave a formula for the finite summation of series using repeated finite integration by which series whose terms are of the form $(ax+b)^{(m)}$, $\phi(x)$, $\alpha^x\phi(x)$, where $\phi(x)$ is a polynomial, can always be summed, and the forms $(ax+b)^{(-m)}$ and $U_x\phi(x)$ can be summed when the necessary number of integrations can be formed.

4. Professor Fleddermann showed there exists uniquely a least super sphere to every bounded set and derived some of the properties of this sphere.

5. A pencil of surfaces of order N containing an $(N-2)$–fold line d in its base can be made to determine a Cremona involution in space by putting its members in projective correspondence with the points of the line d. A point P determines a member S_N of the pencil which corresponds to a unique point z of d. The line joining P and z meets S_N again in P' the inverse of P in the involution. Professor Gentry discussed the particular features of this transformation which arise when the curve residual to d in the base of the pencil becomes composite. It is found that, in addition to lines and conics, this curve may break up into two, three or four parts. Each conic reduces the order of the transformation by 1 and the number of fundamental curves of the second kind by 3. The only other effect of the degeneracy of the base curve is to change the configuration of the fundamental curves of the second kind.

6. Professor Hopkins discussed the equation $f(x)=0$ of degree n with coefficients in a finite field F_p and irreducible over F_p. The quotient-ring $F_p[x]/(f(x))$ is a field $K=F_p(\theta)$ of degree n over F_p, in which not only $f(x)$, but every irreducible polynomial $\phi(x)$ in $F_p[x]$ of degree n, can be decomposed into the product of linear factors. For a fixed prime p one can construct a field

Ω—the 'smallest' algebraically-closed extension of F_p—such that every equation, of arbitrary degree, with coefficients in F_p has all its roots in Ω.

7. Let P_x be the probability that a variate taken at random from a distribution will deviate from the origin by an amount at least as great as x. Assume a frequency function $y=f(x)$ which increases from the origin to a point $[c\sigma, f(c\sigma)]$ and decreases beyond that point. Professor Smith showed that the corresponding probability function $y=P_x$ begins at (0, 1), is concave downward to the deviation $c\sigma$ and concave upward beyond that point. Using the chord as a means of approximating the value of P_x beginning at (0, 1) and extending through $(c\sigma, P_{c\sigma})$ to a point near the distance $c\sigma/(1-P_{c\sigma})$, a former approximation $P_{2\sigma} \leqq .092$ is reduced to $P_{2\sigma} \leqq .056$.

8. Professor Rickey discussed the problem of determining the minimum time required to row a boat to shore and then walk to a designated point down the straight shore line. He showed that this time is not independent of the distance to the point as might be inferred from the results usually obtained in Calculus classes and discussed in detail the various cases involved.

9. Professor Brown computed the temperature distribution in the fins of an air-cooled engine, assumed to be losing heat to the air on two sides and one exposed end, the other end being attached to the engine cylinder. For a simple case, thin fin of rectangular section, this temperature is given by the equation

$$\theta = \theta_h \frac{\cosh a(x - w^1)}{\cosh aw^1}, \quad a = \sqrt{\frac{2q}{Kt}}$$

where q = coefficient of heat transfer from the surface, w = true width of fin, t = fin thickness, x = distance from the cylinder wall, K = fin conductivity, θ = temperature of fin above the air, θ = temperature of cylinder wall above the air, H = heat dissipated by the fin per unit time, H_h = heat dissipated by equal area of wall surface per unit time, w' = corrected fin width $= w+t/2$. If fin effectiveness is defined as the ratio of H to H_h then it was shown to be given by the equation $f = (\tanh aw')/aw'$. The function $(\tanh u)/u$ was discussed briefly.

10. The simple formula $M = 24C/(P(n+1))$ giving the approximate installment rate of interest is found in high school and college texts. Professor Smith showed this formula to be quite accurate, provided the number of monthly installments is not too great. He compared the simple formula to a formula for the installment interest rate derived in his paper. His formula is compact and easily applied. It is derived by finding the compound amounts of the overpaid interest for each month placing these portions of interest at compound interest at the nomial rate j and compounded m times annually.

11. Professor Temple referred to a Wronskian (see "Some functions analogous to the trigonometric and hyperbolic functions" by Temple, *The National Mathematics Magazine*, March, 1939) of order n of solutions of $d^ny/dx^n + y = 0$ which is identically equal to unity, and which is analogous to the identity $\sin^2 x + \cos^2 x \equiv 1$ for $d^2y/dx^2 + y = 0$. He then stated some theorems of unit

identities composed of determinants of orders 2 and 3 where n is a multiple of 2 or 3. He also gave theorems showing that there are zero identities which accompany these unit identities. Proofs of these theorems were not given.

12. Mr. Wicht presented an analytic method of poles and polars by which treatments of remarkable points of the tetrahedron and its cevian are simplified. The notion of linear dependence and one-dimensional homogeneous coordinates were used freely. By this method the theorems of Commandino and Mannheim were outlined. It was shown that the cross ratio of the corresponding vertices of any four consecutive cevians is 4/7. Some of these results were proved synthetically by N. A. Court, this MONTHLY, 1936, p. 89.

W. V. PARKER, *Secretary*

MEETING OF THE NORTHERN CALIFORNIA SECTION

A meeting to organize the Northern California Section of the Mathematical Association of America was held in the Galileo High School, San Francisco, on Saturday, January 28, 1939. A. L. McCarty of the San Francisco Junior College, acting chairman of the Section, presided. Sessions were held both in the morning and in the afternoon, and members and visitors lunched together during the recess.

The attendance at the two sessions was approximately sixty, including the following twelve members of the Association: H. M. Bacon, C. E. Corbin, G. C. Evans, Emma V. Hesse, R. D. James, Sophia H. Levy, A. L. McCarty, F. R. Morris, Falka G. Sturges, Gabriel Szegö, R. K. Wakerling, Harriet A. Welch, and four applicants for membership: T. J. Bass, Jr., Adeline M. Scandrett, Ethel Spearman, Ruth G. Sumner.

The program for the meeting and a proposed set of By-Laws had been prepared by a Joint Program and By-Laws Committee composed of G. C. Evans, University of California; A. L. McCarty, San Francisco Junior College; and Emma V. Hesse, University High School, Oakland. The proposed By-Laws were reported by Professor Evans and adopted by the Section, subject to the approval of the Trustees of the Association. Officers for the coming year were elected as follows: Chairman, A. L. McCarty, San Francisco Junior College; Vice-Chairman, Sophia H. Levy, University of California; Secretary-Treasurer, H. M. Bacon, Stanford University. On presentation of a request that the Section name a person to serve as associate editor of the California Journal of Secondary Education, the Secretary was instructed to cast a ballot for Mrs. Ruth G. Sumner, Oakland High School.

The main part of the program was given over to an address by Professor V. F. Lenzen of the department of physics, University of California. Four other papers were presented. The list of titles and speakers follows:

1. "Some adventures in teaching mathematics to freshmen" by Dr. M. J. Polissar, San Francisco Junior College.

2. "Physical geometry" by Professor V. F. Lenzen, University of California.

3. "Some qualitative properties of the solution of linear differential equations of the second order" by Professor Gabriel Szegö, Stanford University.

4. "Contemporary viewpoints in the teaching of plane geometry" by J. W. Hoge, University High School, Oakland.

5. "The place of mathematics in secondary education" by Adeline M. Scandrett, Mission High School, San Francisco.

Abstracts of the papers follow, numbered in accordance with their listing above:

1. Dr. Polissar sketched a course in problem-solving for freshmen in science courses, in which emphasis is placed on problems which are not stereotyped, but which require a certain amount of intuitive experimentation before a solution is found. Among other devices, puzzles are used, some strictly mathematical and and others requiring only logical analysis. Throughout the course consistent use is made of the slide-rule as a time-saver in solving the ordinary algebraic problems involving numerical data.

2. Professor Lenzen's paper appears in the present issue of this MONTHLY.

3. Professor Szegö called attention to two rather general methods which furnish information about the distribution of the zeros and magnitude of the extrema, respectively, of functions satisfying a linear and homogeneous differential equation of the second order. The first method is due to Sturm (1836), and the second to Sonin (1892). Both methods have been used recently rather frequently, sometimes with slight variations, though their possibilities have not been exhausted. A sketch of the underlying ideas was given and both methods were illustrated by the special case of Bessel's function $J_0(x)$.

4. Mr. Hoge enumerated some of the present day viewpoints of teachers of plane geometry concerning knowledge and use of basal propositions as the major objective of teaching plane geometry, a reduction in the number of required theorems, together with an increase in geometric originals and an increase in postulates, especially at the beginning of the course; integration of the mathematics of arithmetic, algebra, and trigonometry, with plane geometry; attention to patterns of teaching; practical applications used consistently and continuously; development of logical thinking; and teaching for transfer.

5. Miss Scandrett presented extracts from the Report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics on "The Place of Mathematics in Secondary Education." Education aims to develop well-rounded and adaptable individuals. The study of mathematics will develop in individuals the desirable characteristics that achieve these aims provided that the teaching methods are the best. After considering the mathematical needs of many people, the Commission presents a program for mathematics in grades seven to fourteen. To improve our teaching methods, to provide adequate guidance for our pupils, and to revise the curriculum, we must investigate and solve the problems now confronting us in the field of testing.

SOPHIA H. LEVY, *Secretary pro tem.*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The nineteenth regular meeting of the Southern California Section of the Mathematical Association of America was held at Whittier College, Whittier, California, on Saturday, March 4, 1939. Professor W. M. Whyburn, chairman of the Section, presided.

The attendance was seventy, including the following thirty-four members of the Association: L. J. Adams, O. W. Albert, L. D. Ames, Harry Bateman, E. T. Bell, Jessie R. Campbell, P. H. Daus, D. C. Duncan, Harriet E. Glazier, C. G. Jaeger, G. R. Kaelin, G. R. Livingston, Ada McClellan, W. E. Mason, I. P. Maizlish, W. O. Mendenhall, A. D. Michal, P. M. Niersbach, Lena E. Reynolds, J. M. Robb, G. E. F. Sherwood, Marcus Skarstedt, C. E. Smith, D. V. Steed, T. Y. Thomas, W. I. Thompson, C. W. Trigg, S. E. Urner, Morgan Ward, L. E. Wear, D. E. Whelan, Jr., Mabel G. Whiting, W. M. Whyburn, Euphemia R. Worthington.

The following officers were elected for the coming year: Chairman, G. R. Livingston, San Diego State College; Vice-Chairman, O. W. Albert, University of Redlands; Program Committee, Harry Bateman, Chairman, D. C. Duncan, and the Secretary. The next meeting was tentatively scheduled to be held on March 2, 1940, at Compton Junior College.

The following six papers were read.

1. "Mathematics and the integrated curriculum" by Professor Marcus Skarstedt, Whittier College.

2. "Lattice theory and its applications" by R. P. Dilworth, California Institute of Technology, introduced by Professor Ward.

3. "The application of group theory to the normal vibrations of a cubic crystal" by Professor W. V. Houston, Professor of Physics, California Institute of Technology, introduced by Professor Ward.

4. "A new approach to the teaching of algebra and geometry in high schools" by P. M. Niersbach, Bell High School, Huntington Park.

5. "The fundamental sufficiency theorem in the calculus of variations" by Dr. F. A. Valentine, University of California at Los Angeles, introduced by Professor Daus.

6. "A note on the equation $x^y=y^x$" by Dr. F. A. Butter, Jr., University of Southern California, introduced by Professor Steed.

Abstracts of the papers follow, numbered in accordance with their place on the program.

1. To counteract the tendency in most small liberal arts colleges to eliminate mathematics as a required subject for graduation, various general courses called survey courses or integration courses or appreciation courses have been set up, and a number of textbooks for such courses have been written. These range in presentation and subject matter from the very simple to the almost profound. Professor Skarstedt outlined a special plan of integrated courses now in operation at Whittier College, and discussed especially the part devoted to mathe-

matics which is not so much concerned with teaching mathematics as teaching about mathematics. It is required of all freshmen, but the ordinary well-recognized courses in mathematics are offered for those whose main interests lie in mathematics and the physical sciences.

2. Mr. Dilworth began with a short historical account of the development of lattice theory, emphasizing the manner in which the lattice-theoretic ideas arose in number theory, set theory, geometry, logic, *etc.* Lattices with a multiplication were described and a short account was given of their applications to algebra, particularly to the Noether decomposition theorems for the ideals of a commutative ring.

3. Professor Houston illustrated the use that can be made of group theoretical considerations in analyzing the normal vibrations of a crystal in the case of a simple cubic lattice. The group under which the system is invariant can be made finite by the use of periodic boundary conditions. The invariant sub-group that contains the translations only can be completely reduced on the basis of plane waves of arbitrary polarization. These can be divided into sets of 144 members, and each set gives rise to a representation of the whole group of the crystal. In a number of special cases these representations can be reduced with respect to various sub-groups of rotations to such an extent that it is possible to determine the directions of the normal vibrations. In such cases the frequencies can be directly expressed in terms of the force constants. The group theoretical treatment makes it easy to distinguish those properties of the elastic spectrum which depend only on the symmetry of the lattice from those which depend on special properties of the force constants.

4. Mr. Niersbach described the work of G. R. Mirick of the Lincoln School, Teachers' College, Columbia University, who during the last five years has been experimenting in a new method of teaching algebra and geometry suggested by Dr. Veblen in a talk on the modern approach to elementary geometry made at the Rice Institute in 1932. He is working on a course in which algebra and geometry are taught simultaneously. The analytic method, which is so important in other branches of mathematics and science, is introduced. While it is a complete change from the traditional methods of Euclid, this method incorporates an understanding of the use of postulational thinking, the development of the power of analysis, and an appreciation of the nature of proof.

5. The fundamental sufficiency theorem in the calculus of variations can often be used to obtain a minimum for a definite integral in a prescribed region. Dr. Valentine described a method of prescribing the region so that it could be simply covered by a field of extremals. The construction was made for the case when the Euler equation has a first integral.

6. Dr. Butter considered the equation $x^y=y^x$, $(x>0,\ y>0)$. It was shown that if $0<x\leqq 1$, or if $x=e$, then $y=x$. On the other hand, if $0<x<e$, or if $x>e$, then two values of y are determined: the one $y=x$; the other $y>e$, or $1<y<e$, respectively. It was shown that if $y\neq x$, then $y\rightarrow 1+0$ as $x\rightarrow+\infty$, and $y'(x)<0$. The graph of the equation was also given.

P. H. DAUS, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The twenty-fourth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 8, 1939, with a morning session, a dinner, and an afternoon session. Professor C. O. Williamson, chairman of the Section presided at these sessions. As a happy addition to the program Professor G. A. Bliss of the University of Chicago was guest-speaker.

Seventy persons registered attendance, including the following forty-one members of the Association: G. E. Albert, W. E. Anderson, F. R. Bamforth, Grace M. Bareis, I. A. Barnett, H. M. Beatty, H. A. Bender, G. A. Bliss, Henry Blumberg, M. G. Boyce, J. B. Brandeberry, O. E. Brown, R. S. Burington, Rufus Crane, Wayne Dancer, O. L. Dustheimer, T. M. Focke, N. A. Gilbert, B. C. Glover, E. M. Justin, L. C. Knight, A. C. Ladner, Jesse Pierce, H. L. Pollard, D. W. Pugsley, Tibor Radó, C. E. Rhodes, Hortense Rickard, R. F. Rinehart, N. S. Risley, S. A. Rowland, G. W. Starcher, H. E. Stelson, Otto Szász, C. F. Thomas, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, C. R. Wylie, Jr., C. H. Yeaton.

The following officers were elected for the coming year: Chairman, Wayne Dancer, University of Toledo; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, J. H. Weaver, Ohio State University; Member of Program Committee, I. A. Barnett, University of Cincinnati. It is expected that the next meeting will be held at the Ohio State University, April 4 or 6, 1940.

The following eight papers were read:

1. "Use of conformal mapping in shaping wing profiles" by Professor R. S. Burington, Case School of Applied Science.

2. "The semi-regular solids" by Professor Wayne Dancer, University of Toledo.

3. "Some properties of pairs of circular cubics associated with four points in a plane" by Professor M. G. Boyce, Western Reserve University.

4. "Extrema of k-symmetric functions of n variables which are connected by $n-k+1$ k-symmetric relations" by Dr. R. F. Rinehart, Case School of Applied Science.

5. "A certain minimum problem associated with parallelograms" by Professor I. A. Barnett, University of Cincinnati, and Dr. Paul Pepper, University of Notre Dame, presented by Professor Barnett.

6. "An arithmetical property of the cosine function" by Dr. Otto Szàsz, University of Cincinnati.

7. "Why I teach mathematics" by the chairman of the Section, Professor C. O. Williamson, College of Wooster.

8. "The Hamilton-Jacobi theory in the calculus of variations and its sources" by Professor G. A. Bliss, University of Chicago, by invitation of the committee.

Abstracts of these papers follow:

1. Professor Burington discussed the application of conformal mapping to the problem of shaping airplane wing profiles, using slides to assist in the presentation. The paper was especially timely due to the fact that the growing improvement in the efficiency of airplanes has renewed interest in the question of shaping their contours mathematically. A brief summary of the airfoil theories of Joukowski, Karman-Trefftz, and Mises was given, together with a short treatment of the family of airfoils recently advanced by Piercy. A discussion of the advantages and limitations of these theories was given. The treatment was confined to the ideal two-dimensional case.

2. After defining the semi-regular solids, Professor Dancer explained the relationship between these figures and the five regular polyhedra. The two types, the facially regular and the vertically regular, were used to illustrate the principle of duality. The speaker showed why there can be no more than thirteen polyhedra in each of the two classes. The star polyhedra were also defined in terms of the simpler solids. Models of all the solids discussed were presented.

3. Two types of loci of a variable point P relative to four fixed points A, B, C, D were discussed by Professor Boyce. The first locus was defined by the condition that the angles APB and CPD be equal (modulo π) and the second that the ratios AP/BP and CP/DP be equal. These loci, both being circular cubics of the type called focal curves, have long been known and were extensively studied by Van Rees, Teixeira, and others. In this paper, however, the use of complex coördinates with the Argand diagram as a reference system yielded shorter proofs for many of the known properties and some new results, especially with reference to the mutual relationships of the two kinds of loci.

4. Dr. Rinehart discussed an extension of two well-known algebraic theorems: I. The function $x_1x_2 \cdots x_n$, where the variables are subject to the condition $x_1+x_2+ \cdots +x_n=K>0$ has a relative maximum at $x_1=x_2= \cdots =x_n$. II. The function $x_1+x_2+ \cdots +x_n$, where the variables are subject to the condition $x_1x_2 \cdots x_n=K>0$, has a relative minimum at $x_1=x_2= \cdots =x_n$. It was shown that these results are special cases of a very general theorem, a loose statement of which is: A function $f(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n)$ which is symmetric in $x_1, \cdots, x_{k+1}$, where the variables satisfy the $n-k$ functional equations $V^{(i)}(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n)=0$, $(i=1, \cdots, n-k)$, where each $V^{(i)}$ is symmetric in $x_1, \cdots, x_{k+1}$, has a relative maximum or minimum at $x_1=x_2= \cdots =x_{k+1}$, according as a certain determinant is negative or positive.

5. If a parallelogram with fixed sides a and b and included angle θ is given, there are an infinite number of parallelograms with one side of length c that may be inscribed in the given parallelogram. Professor Barnett presented the problem of finding those inscribed parallelograms which have minimum perimeter. The problem reduces to finding the shortest distance from a point to an ellipse. A discussion of the dependence of this minimum perimeter on the quantities a, b, θ, and c was given. The particular cases where the given paral-

lelogram is a rectangle or a rhombus are of interest since in these cases the minimum inscribed parallelogram may be constructed with straight edge and compases only.

6. In a joint paper by Professor Szàsz and Professor Barnett, there arose the problem of determining all real rational values of x for which $\cos \pi x$ is rational, or is a quadratic irrationality. With the non-essential restriction $0<x<1$, we find that the only rational values of $\cos \pi x$ are 0, $\pm 1/2$, with 1/2, 1/3, 2/3 as the corresponding values of x. In the irrational case, $\cos \pi x$ must be $\pm\sqrt{2}/2$, $\pm\sqrt{3}/2$, $(1\pm\sqrt{5})/4$, $(-1\pm\sqrt{5})4$, where the corresponding values of x are respectively 1/4, 3/4, 1/6, 5/6, 1/5, 3/5, 2/5, 4/5. It is noticeable that $\cos \pi x$ is algebraic whenever x is rational, and it is transcendental if x is an algebraic irrationality.

7. Archimedes said "Give me a place on which to put my fulcrum, and I can move the earth." It was the thesis of Professor Williamson's paper that mathematics, set in the solid concrete of observed facts, is the fulcrum which Archimedes needed, and that by using observed data as the power on one end of the lever we are able to turn over and pry out new facts, and thus move the world. Newton moved the world of thought onto a new plane by this process. Our students need to know the principles of this process.

8. Professor Bliss's paper was concerned with a chapter in the calculus of variations which is called the Hamilton-Jacobi theory. It received this name because in the first half of the last century these two fertile-minded mathematicians were the most active in the formulation and application of the theory, especially in problems of dynamics. But the sources of the chapter lie in the beginnings of the calculus of variations in the first years of the eighteenth century and even earlier. Since then the theory has become a permanent and important part of classical dynamics, with applications especially in the perturbation theories of celestial mechanics. More recently it has greatly influenced the development of the calculus of variations and its applications to geometry as well as mechanics. The purpose of the present paper is to show how this evolution occurred, especially as a result of the comparison of problems in optics and dynamics. The paper was suggested by studies of recent formulations of the Hamilton-Jacobi theory for parametric problems in the calculus of variations by Carathéodory and Teach.

RUFUS CRANE, *Secretary*

THE FALL MEETING OF THE MINNESOTA SECTION

A second meeting in 1938 of the Minnesota Section of the Mathematical Association of America was held at the University of Minnesota, Minneapolis, Minnesota, on October 28. This was a joint meeting with the Mathematics Section of the Minnesota Education Association.

The general meeting was attended by over two hundred representatives of the colleges and secondary schools of the state, including the following thirty-five members of the Association: Jacob Bearman, C. J. Blackall, W. E. Brooke, L. E. Bush, W. H. Bussey, E. J. Camp, S. Elizabeth Carlson, Sister M. Claudette, H. H. Dalaker, J. H. Daoust, H. C. T. Eggers, Margaret C. Eide, Gladys Gibbens, C. H. Gingrich, W. L. Hart, W. N. Herr, J. S. Hickman, Dunham Jackson, R. E. Langer, J. D. Novak, Lois E. Pollard, Inez Rundstrom, J. M. Rysgaard, R. B. Saunders, M. G. Scherberg, Ole Schey, C. Grace Shover, A. J. Strane, F. J. Taylor, H. P. Thielman, Ella Thorp, Robert Tucker, A. L. Underhill, K. W. Wegner, Marion A. Wilder.

The meeting started with a luncheon attended by one hundred forty-five persons. Following the luncheon, Dr. W. E. Peik, Dean of the College of Education, University of Minnesota, gave an address. At the general meeting in the afternoon Mr. W. B. Gundlach, Rochester High School, and Professor W. H. Bussey, University of Minnesota, presided.

The program consisted of the following three papers:

1. "The use of National Council publications as reference material for mathematics teachers" by Edith Woolsey, Sanford Junior High School, Minneapolis.

2. "People and mathematics" by Professor Dunham Jackson, University of Minnesota.

3. "Josiah Willard Gibbs, an American mathematician" by Professor R. E. Langer, University of Wisconsin.

A. L. UNDERHILL, *Secretary*

THE ANNUAL MEETING OF THE TEXAS SECTION

The 1939 annual meeting of the Texas Section of the Mathematical Association of America was held in Abilene on Friday afternoon, March 31, and Saturday morning, April 1. Abilene Christian College, Hardin-Simmons University, and McMurray College acted cooperatively as hosts for the meeting. The Section chairman, Professor H. J. Ettlinger, presided at both sessions.

Among the fifty-two people attending the meeting were the following nineteen members of the Association: H. E. Bray, J. E. Burnam, Nat Edmonson, Jr., H. J. Ettlinger, C. A. Gilley, E. H. Hanson, E. R. Heineman, G. B. Huff, H. A. Luther, B. C. Moore, E. D. Mouzon, Jr., M. E. Mullins, C. A. Murray C. R. Sherer, F. W. Sparks, Jennie L. Tate, Earl Thomas, F. E. Ulrich, H. E. Woodward.

Those attending the meeting were the guests of the host schools at a dinner on Friday evening. At the business session the following officers were elected for the coming year: Chairman, E. D. Mouzon, Jr., Southern Methodist University; Vice-Chairman, J. E. Burnam, Hardin-Simmons University. The 1940 meeting was awarded to Southern Methodist University, Dallas, Texas. The fixing of the date for the 1940 meeting was left to the Section officers.

The following papers were read:

1. "The problem of type for Riemann surfaces" by Dr. F. E. Ulrich, The Rice Institute.

2. "On the consequences of the continuum hypothesis" by Walter Jennings, A. and M. College of Texas, introduced by Professor Edmonson.

3. "The evaluation of a binomial determinant" by Professor E. R. Heineman, Texas Technological College.

4. "A new approach to the Hermite polynomials" by Robert Greenwood, University of Texas, introduced by Professor Ettlinger.

5. "The development of the problem of Bolza in the calculus of variations" by Dr. C. P. Brady, Texas Technological College, introduced by Professor Sparks.

6. "The Bartky method of solving linear matrix equations" by Professor H. J. Ettlinger, University of Texas.

7. "Regular curve families in the plane" by Wilfred Kaplan, The Rice Institute, introduced by the Secretary.

8. "Computations for three dimensional seismographic shooting" by Dr. J. T. Hurt, A. and M. College of Texas, introduced by Professor Edmonson.

9. "Reducing the number of failures in freshman mathematics" by Professor C. A. Murray, West Texas State Teachers College.

Abstracts of some of these papers follow, numbered in accordance with their place on the program:

1. Dr. Ulrich discussed the deficiency relation of Nevanlinna for meromorphic functions and indicated its significance in the study of the Riemann surface on which the smooth plane is mapped by the function. He gave the function cos z as an example of a function with positive total deficiency and positive total ramification. He formulated the problem of type and stated the theorems of Picard, Bloch and Gross. Finally he gave the criterion of Ahlfors for the parabolic type, and for certain classes of surfaces showed how by a suitable choice of metric this criterion can be used to determine sufficient conditions for the parabolic case.

3. Professor Heineman indicated a convenient method of expansion for the determinant whose $(i+1)$th row is $\left| a^{nl_i}, a^{(n-1)l_i}\ a^{(n-2)l_i}\ b^{2l_i}, \cdots, b^{nl_i} \right|$, where $i=0, 1, 2, \cdots, n-1$, and $l_{i+1}>l_i$. The value of this determinant was expressed in terms of $(b-a)$, ab, and the symmetric functions $F_r=\sum_{i=0}^{r} a^{r-i}b^i$.

4. Mr. Greenwood's paper presented material taken from the paper of the same title by E. U. Condon and Robert Greenwood, *Philosophical Magazine*, Series 7, vol. XXIV, 1937, page 281.

5. Bliss formulated the problem of Bolza and discussed necessary conditions; Hestenes was the first to formulate a condition of Mayer by the use of which a sufficiency proof could be made; following Hestenes's sufficiency theorems Bliss, Reid, Morse, and Hestenes proposed other useful forms of the Mayer condition. Dr. Brady in his Chicago thesis made use of the theory of the problem of Bolza to study the problem of minimizing a function of integrals, which as formulated by Dr. Brady is readily transformed into a problem of Bolza. However, an application of the sufficiency theorems for the latter problem yields only a restricted strong relative minimum for the former because the definitions of a strong relative minimum for the two problems are not equivalent. A satisfactory theorem concerning strong relative minima was deduced by the use of supplementary methods similar to those used by Hestenes for isoperimetric problems. The fact that the theory for the problem of Bolza failed to yield the results desired for the isoperimetric problem and for Dr. Brady's problem led Hestenes last fall to a reformulation of the problem of Bolza.

6. Professor Ettlinger discussed the matrix exponential functions obtained by Bartky. These methods, though completely independent of Heaviside's work, are nevertheless analogous to the latter's operational methods. In general these matrix exponential functions do not satisfy the ordinary exponential laws. The principal result presented in this paper is that, if the group of exponentials belonging to matrices having the same multiplicities for their characteristic numbers be put in a canonical form, the resulting exponentials obey the ordinary laws of exponents.

7. Mr. Kaplan considered families of Jordan curves $x=x(t)$, $y=y(t)$ in the plane such that (a) through each point of the plane passes exactly one curve, (b) locally the curves behave like parallel lines. He then established that each curve is open and tends to infinity in both directions. On the basis of the manner in which each triple of curves of the family divides the plane, the family can be pictured as an abstract system with two triadic order relations, termed a normal bracket system. To every such abstract system corresponds a curve family which generates it. Two curve families have the same normal bracket system if and only if one is a one-to-one continuous image of the other.

8. Dr. Hurt obtained equations for three dimensional seismograph reflection shooting, and put them into a form suitable for routine use. The resulting graphs and tables are universal in scope because the solutions are exact, no approximations are made beyond the initial assumptions, and they apply in any area since no knowledge of the variation of velocity with depth is assumed.

NAT EDMONSON, Jr., *Secretary*

PHYSICAL GEOMETRY

V. F. LENZEN, University of California, Berkeley

1. Introduction. Prior to Einstein a distinction was usually made between geometry and physics. Geometry was viewed as a rational science which is independent of sensory experience; physics was known to be an empirical science based upon observation and experiment. The sharp separation between mathematics and physics may be illustrated by the sciences of kinematics and dynamics. In his *Principles of Mechanics*, which was published in 1905, Slate says, "In the first two chapters we shall be occupied with conceptions—Velocity and Acceleration—that rest entirely upon a mathematical basis. . . . If mechanics is taken to include kinematics also, as it frequently is, that part of the science which is physical and not geometrical must be specially distinguished. It is designated as Dynamics. The point should be watched at which the transition is · · · made by introducing experimental results into the framework of our science." The ideas expressed by Slate are characteristic of older books on mechanics. In the study of motion there was recognized the progression: geometry, the science of space; kinematics, the science of motion which was based upon the addition of time to space; dynamics or mechanics, which explained the motions of the material bodies in the physical world. Geometry and kinematics were viewed as mathematical sciences, dynamics or mechanics as a physical science. In the present paper I shall show how geometry and physics have been united in the science of physical geometry.

2. Historical sketch. Our discussion of the relation of geometry to physics may well be prefaced by a description of its subject matter. Geometry is frequently defined as the science of space, but what is space? One of the best answers to this question is given in Carnap's early monograph, *Der Raum* [1]. In this work he distinguishes between formal space, intuitional space, and physical space. Formal space is a system of general ordinal relations. The formal properties of the terms and relations of such a structure are determined by postulates. Formal or abstract space is the subject matter of abstract geometry. Intuitional space is the system of relations between spatial objects such as lines, surfaces, and volumes, the properties of which are apprehended in sense-perception or imagination. Intuitional space is especially considered in the Kantian philosophy of geometry. Physical space is the system of relations between the bodies and phenomena of the physical world and is the subject matter of physical geometry. It may be added that the distinction between topological, projective, and metrical properties applies equally to formal space, intuitional space, and physical space. The present discussion will find need only for abstract geometry and physical geometry.

The development of an understanding of the relation between geometry and physics may be credited principally to the theory of relativity. This theory initiated a program for the reduction of physics to geometry. The special theory

of relativity made it possible to express kinematics in terms of a four-dimensional space-time. In the general theory, space-time is viewed as a Riemannian continuum whose curvature is determined by matter. A free material particle describes a world line which is a geodesic of this continuum. The general theory of relativity thus reduces the physics of gravitation to geometry, and unified field theories have been constructed in order to reduce all physics to geometry. This geometrization of physics appears to have made it a branch of mathematics, to have freed it from dependence on experience. A unified mathematical representation of physical phenomena is offered, and this achievement has inspired Sir James Jeans to declare that God is to be conceived as a pure mathematician.

The reduction of physics to geometry requires, however, that geometry be exhibited as an empirical science. In so far as geometry can be applied to the physical world it is based upon observation and experiment. I shall represent geometry to be the most firmly established branch of physics. If physics is to be reduced to geometry, geometry must also be reduced to physics.

That the concept of physical geometry is a significant contribution may be shown by exhibiting historical philosophical interpretations of geometry. Geometry as a mathematical science was created by the ancient Greeks, but the raw materials for a geometry were fashioned by their predecessors, notably the Egyptians. The Egyptians had to make surveys of land in order to redetermine the marks of boundaries which had been washed away by the floods of the Nile. Hence they measured distances and lengths and discovered propositions that express the metrical relations of the elements of simple figures. The Egyptians thus discovered and used propositions of physical geometry. The Greeks organized such propositions into a deductive science; Euclid founded geometry upon axioms and postulates from which propositions may be derived as theorems. Euclidean geometry has furnished the classical model for science.

The Greeks created the deductive science of geometry and originated the view that geometry is a rational science which is independent of sensory experience. Thus Plato taught that the objects of science must be universal and permanent. The objects of perception are in a state of flux, and hence propositions about the world of experience are infected with uncertainty and relativity. He explained the possibility of rational science by the theory of a transcendent world of pure forms, or ideas, which can be known only by reason. Geometrical structures such as triangles and circles are pure forms which are to be distinguished from the crude perceptible triangles and circles in the world of sense-perception. Geometry is approximately applicable to experience because perceptible figures participate in the pure forms. The soul has direct knowledge of pure forms in a pre-earthly state of existence; perception through the senses stimulates recollection of the pure forms in which the objects of perception participate. In support of his theory that knowledge of geometrical figures is latent in the individual mind, Plato narrates how Socrates guides an uneducated slave boy step by step to the recognition of the truth of a proposition in ge-

ometry. Thus the Platonic philosophy of geometry interpreted the objects of geometry to be ideal entities which transcend ordinary experience.

Since the eighteenth century the theory of Kant has exerted a widespread influence. Kant started from the assumption that pure mathematics, which is exemplified by geometry, is *a priori* and therefore independent of experience. He propounded the question, how is pure mathematics possible? His answer as applied to geometry was that space is the *a priori* form of external intuition which is the condition of all perceptual experience. Geometrical figures are constructions in space and can be constructed in pure intuition independently of sensory experience. This theory provided a new foundation for the interpretation of geometry as the science of universal and necessary truths.

The Kantian theory dominated the philosophy of geometry during the nineteenth century. Geometrical figures were assumed to be constructed in pure intuition and analysis of such figures yielded the self-evident axioms of Euclidean geometry. In recent years the German philosopher Husserl has offered intuitions into the essence of geometrical structures as the foundation of geometry. Intuitional space which is referred to by Carnap, is an inheritance from Kant. During the nineteenth century, however, the non-Euclidean geometries were created and led to the development of new points of view. Helmholtz and others exhibited intuitive models of the non-Euclidean geometries, and thus shook the Kantian doctrine that intuition reveals physical space to be Euclidean. The study of foundations led to the abstract theory of geometry, according to which the propositions of geometry are blank forms devoid of empirical reference. The postulates of a geometry constitute an implicit definition of the fundamental concepts which express the properties of formal space. Geometrical theory is concerned with the deductive dependence of theorems upon postulates. Since postulates and theorems are devoid of empirical significance, the problem of their truth or falsity does not arise. A proposition in geometry becomes true or false only when a concrete interpretation is given to the concepts.

The criticism of the theory that pure intuition is the origin of geometry was accompanied by the development of the view that in so far as geometry can be used in physics, geometrical propositions express the positional relations of perceptible bodies. Gauss measured the angles of a physical triangle whose sides were light rays, in order to test whether or not the sum of the angles is equal to two right angles. Helmholtz [**2**] in an essay on the origin and significance of the axioms of geometry declared that these axioms describe the mechanical behavior of our most rigid bodies during motions. Riemann [**3**] in his famous essay on the hypotheses which constitute the foundations of geometry advanced the hypothesis that the metrical structure of physical space depends on the physical forces in it. Thus the question, is physical space Euclidean or non-Euclidean?, acquired significance. The significance of this question presupposes that the metrical structure of space is defined in terms of the positional relations of physical bodies or phenomena. The standpoints of Gauss, Helm-

holtz and Riemann eventually were realized in the contemporary concept of physical geometry which is exemplified in Einstein's relativistic theory of gravitation. Geometry, in so far as it is relevant to physics, is a physical science that is based upon observation and experiment.

3. An operational theory. Synthetic treatment. The function of physical geometry is to describe the properties of physical space. In preparation for an exposition of how physical geometry may be developed, it is desirable to set forth the elements of the problem. In agreement with Carnap, I distinguish data of experience, postulate of measure, and relational structure. Data of experience are the contact of two points at a specific time, the incidence of a point on a line, the inclusion of a body by a surface, and so forth. Perceptions of contact, or of coincidence, especially furnish the raw materials of geometry. But such data of experience are sufficient only for the topological structure of space. Projective properties require the determination of straight lines, and metrical properties require procedures for measuring length and angles.

Projective and metrical geometry are relative to definitions which are matters of convention. Carnap has clearly shown that it is possible to proceed in two ways. One may adopt a postulate of measure and then by observation determine the scheme of geometrical relations that describes the metrical structure of space. Experience determines whether physical space is Euclidean or non-Euclidean only if a standard of measure has been adopted. It has been traditional to adopt as standard of measure the distance between two points on a rigid body and to postulate that this distance is independent of position. As Carnap has pointed out, an alternative procedure is to postulate the scheme of geometrical relations and then determine from experience the standard of measure that is implied. The possibility of this procedure was especially emphasized by Poincaré, who declared that geometry is determined by conventional definitions. He contended that since Euclidean geometry is the simplest, convention will decree its continued employment for the description of physical phenomena. If light did not travel in straight lines, Euclidean geometry could still be used to formulate different laws of physics. The general theory of relativity, however, predicts a behavior of rigid bodies which makes it convenient to change the geometry rather than the standard of measure.

The foregoing discussion demonstrates that metrical physical geometry exemplifies the operational theory of physical concepts. This theory, which has been expounded notably by Bridgman [4], expresses the meaning of physical concepts in terms of operations. In order to measure a physical quantity it is necessary to control the conditions under which a quantity assumes a determinate value. The procedures of measurement require physical and mental operations that are performed in accordance with prescribed rules. The definition of a physical quantity is expressed by the description of the conditions and procedures of measurement. Consistent application of this operational theory leads to the interpretation of a physical quantity as a number assigned to a

physical property of bodies. Thus the definition of a physical quantity does not express an intuitive insight into an intrinsic essence of the quantity. Textbooks of physics have defined mass as the quantity of matter in a body, but this is only a verbal definition. A significant definition of mass must describe the procedure for measuring the mass of a body. The same point of view applies to physical geometry. Consider, for example, the concept of length. Some philosophers have declared that we have a direct perception of length which acquaints us with the meaning of the concept; this has been the basis for the concept of intuitional space. The operational theory, however, recognizes that length as a physical quantity depends on operations of measurement in terms of a standard. The operational nature of length is especially demonstrated in the special theory of relativity.

I have several times referred to a standard of measure as a basis for metrical physical geometry. This standard is based upon the properties of practically rigid bodies. I assume that we are acquainted with examples of such bodies: sticks, stones, and manufactured bodies, such as iron rods. In order to describe the properties of rigid bodies, let us suppose that two points have been made on such a body. A point will be a hole made by a pin or a dot with a pencil. The two points may be called a rigid point-pair and determine a stretch. Given two rigid point-pairs that may be placed alongside each other so that the points of one are in contact with the points of the other. If the rigid pairs are displaced together, the contacts are preserved. If one rigid pair is kept fixed and the other displaced and returned to its initial position, the contacts are restored. If a number of rigid point-pairs can be brought consecutively into contact with a specific pair, they can be brought into contact with one another. Stretches defined by rigid point-pairs in contact are said to be congruent. If it is postulated that the length of a stretch is independent of position, stretches at a distance may be defined to be congruent.

I shall now explain how metrical, physical geometry may be developed so as to describe the properties of physical space. Physical space may be defined as the system of positional relations of perceptible bodies and phenomena. Such positional relations may be investigated from the standpoint of topology, but I propose to study the metrical structure of space. For this purpose we adopt rigid point-pairs as standards of measure. Thus the metrical structure is determined by the positional relations of practically rigid bodies. Indeed, Einstein [5] has described space as the totality of possibilities of relative position of practically rigid bodies.

On investigating the properties of space it is necessary to specify a frame of reference relative to which rigid bodies are at rest or in motion. In elementary geometry a geometrical structure is ordinarily assumed to be at rest in a frame that is rigidly attached to the earth. As we shall see, however, the special theory of relativity has brought to light the relativity of space to a frame of reference.

The procedure in building physical geometry is exemplified by some ele-

mentary experiments which have been described by Carnap [**1**, p. 41]. Let us have given a standard body of which two points A, B determine a standard stretch. Consider a physical surface such as the top of a desk.

(1) We discover that A and B and also C and D of the standard body may be brought simultaneously into contact with four points A_1, B_1, C_1, D_1, upon the surface. Repeated experiments demonstrate that whenever A, B, C or A, C, D or B, C, D are in contact with their corresponding points, the fourth pair of points is in contact. The pair A, B can be brought into contact with B_1 and C_1, with C_1 and D_1, and with D_1 and B_1. The conclusion is that with respect to the point-pair (A, B) as a standard, $A_1, B_1; B_1, C_1; C_1, D_1; D_1, B_1$ are rigid point-pairs. From the first experiment one infers the rigidity of C, D and further the rigidity of the set A, B, C, D.

(2) If A, B, C, D are brought into contact with four other points of the surface, repeated experiments yield the same results as before, and therefore the other points constitute a rigid set of points. All sets of four points of the surface are demonstrated to be rigid, and hence the whole surface is rigid.

(3) In the first experiment it was found that the contact of three pairs chosen from AA_1, BB_1, CC_1, DD_1, involved that of the fourth, provided the fourth was not CC_1. We discover that while A, B, D remain in contact with the corresponding points, an initial contact of C with C_1 may be interrupted. We then declare that A, B, C, D and A_1, B_1, C_1, D_1 have moved with respect to each other, and during the motion three pairs of points have remained in contact. This is the characteristic of a straight line. A, B, D lie on a straight line and so do A_1, B_1, D_1. A straight line is thus defined by point-pairs that remain fixed with respect to a rigid frame during a rotation about the line.

(4) If we bring A into contact with A_1 and simultaneously B in contact with B_1', B_1'', $\cdots$ one after another, it never occurs that D is not in contact with a point of the surface D_1', D_1'', $\cdots$. The points A_1, B_1', D_1', lie on a straight line, also A_1, B_1'', D_1'', and so forth.

(5) If the preceding experiment is performed with A in contact with A_2, A_3, *etc.*, the same results are obtained. Thus from every point in the surface there extend straight lines in the surface in all directions, and hence the surface is judged to be a plane.

As a result of the preceding experiments we have learned how to recognize a straight line and a plane. In practice we test the straightness of a line by the physical law that light travels through a homogeneous medium in straight lines. Straight lines are exemplified by the edge of a solid, by a stretched cord, and by the path of a ray of light.

Our next task is to introduce the concept of distance or length. Suppose that we have given two stretches determined by rigid point-pairs (A,B) and (A_1,B_1) respectively, so that A is in contact with A_1 and B is in contact with B_1. As previously stated, the stretches are said to be congruent. The same length, or distance between their end points, is assigned to each of the congruent stretches. Congruence is directly tested when corresponding points are in con-

tact, but this test fails when the stretches are separated. However, we shall assign the same length, or distance, to the separated stretches. We thus adopt the fundamental postulate that the length of a stretch determined by a rigid point-pair, or the distance between the two points, is invariant in displacement. It is assumed, however, that the temperature remains constant. A standard stretch may be assigned the length one. The length of any straight line can then be determined with respect to our standard. We may measure the length of a line by counting the number of times that the standard can be laid off on the line, or by counting the number of equal stretches that may be placed end to end along the line.

The operational significance of the concept of length is especially exemplified by the special theory of relativity. I have already stated that space is associated with some frame of reference. A fundamental assumption of classical kinematics was that space is absolute, that is, the same for all frames of reference regardless of their state of motion. This means that the geometrical properties of figures were viewed as invariant under a transformation of the frame of reference. Thus the length of a rigid rod was postulated to be the same relative to frames of reference in relative motion with respect to one another. Indeed, it appears to be self evident that the length of a rod represents an intrinsic property which does not depend on the frame of reference. According to the operational theory, however, the concept of length is defined by the method of measurement, and in relativistic theory the result depends on the state of motion of the frame. If the frame is one in which the rod is at rest, an observer can measure the length of the rod in terms of a standard of length by placing a calibrated scale of length adjacent to the rod under investigation and observing the points on the scale that coincide with the end points of the rod. But in a frame relative to which the rod is moving, this procedure is not possible because of relative motion between the rod and the instrument of measurement. A possible procedure is to mark the simultaneous positions of the end points of the rod on the frame of reference. One may then at one's leisure measure the distance between the two points on the frame of reference with a scale at rest. Simultaneity, however, is relative to the frame of reference, and hence the outcome of measuring length is relative. In general, in the theory of relativity the geometrical structure of a body is relative to the frame. A configuration which is described as a circle from a frame relative to which it is at rest is described as an ellipse from a frame relative to which it is moving.

Let us now return to the problem of constructing a physical geometry for structures at rest in a selected frame of reference. We have a standard of length and methods for the recognition of straight lines and planes. We may verify the proposition that a straight line is shorter than an adjacent line between the same points; this proposition may be used to define a straight line. We may construct figures out of straight lines. The properties of a plane triangle may be used to determine the curvature of the plane; the curvature is zero, negative, or positive according as the sum of the angles is equal to, less than, or greater

than two right angles. The curvature of three mutually perpendicular planes at a point determines the curvature of space at that point.

By such procedures we build up a concept of metrical physical space. The positional relations of rigid bodies which determine the metrical structure of space are described by a geometry which is a branch of physics. Applied to the physical world of experience, our procedures yield the result that to the first approximation, at least, actual physical space is Euclidean. The sum of the angles of large triangles, the sides of which are the paths of light rays, is two right angles. It is possible to construct a Cartesian coördinate system out of equal rods. This means that out of a set of rods, the corresponding end points of which coincide when the rods are placed adjacent to one another, it is possible to construct a cubical lattice which is the physical realization of a Cartesian coördinate system.

The propositions that characterize the positional properties of configurations of rigid bodies are only approximately verified by experience on account of lack of precision in observation. In the development of geometry, the fiction of a precise observation is adopted and the propositions are interpreted to express definite relations between definite properties. This procedure makes it possible to study the deductive relations between propositions, and Euclidean geometry may then be founded on axioms which express the properties of a set of terms and relations. We may then transform these axioms into a set of postulates which implicitly define the formal properties of the objects of geometry and thereby obtain an abstract geometry. The structures of physical geometry then exemplify approximately the formal properties defined by the postulates. *In the passage from physical to abstract geometry it does not appear to be necessary to interpolate a science that is founded on pure intuition.*

4. Analytic treatment. In the foregoing discussion I have employed the synthetic method of building geometry, but one may use the analytic method. A Euclidean space is characterized by the fact that it admits a cubical lattice which will serve as a Cartesian coördinate system. The metrical structure of space is described by the formula which expresses the differential element of distance between two points in terms of the differences in the coördinates of the points. Thus Euclidean space admits a Cartesian coördinate system for which

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Curvilinear coördinates may also be used, but the formula for the line element in such coördinates can always be transformed to the Cartesian form.

On a curved surface it is impossible to extend a Cartesian coördinate system over a finite region. Accordingly one introduces Gaussian, that is, curvilinear, coördinates. The coördinate lines may be labelled $u_1=\text{constant}$ and $u_2=\text{constant}$. The position of a point on the surface is specified by giving its Gaussian coördinates u_1 and u_2. The distance between two points u_1, u_2 and u_1+du_1, u_2+du_2 is expressed by the formula

$$ds^2 = g_{11}du_1^2 + 2g_{12}du_1du_2 + g_{22}du_2^2 .$$

The g's are function of u_1 and u_2 and are called the components of the fundamental metrical tensor. The measure of curvature of the surface is a function of the g's and their derivatives.

In the classical accounts of differential geometry the curved surface is viewed as imbedded in a three-dimensional Euclidean space. The g's are expressed as functions of the derivatives of the Cartesian coördinates with respect to the Gaussian coördinates on the surface u_1, u_2,

$$g_{ik} = \frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_k} + \frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_k} + \frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_k} .$$

A physicist, however, prefers an exposition of the immediate physical significance of the g's. The discussion presupposes that in an infinitesimal region the surface may be assumed plane. I assume that we have a standard of length which is invariant during displacements on the surface. ds is the length of the element of arc between the two points relative to the standard. du_1 and du_2 are increments of coördinates and have no immediate metrical significance. As we pass from u_1, u_2 to u_1+du_1, u_2, the distance ds is related to the coördinate increment du_1 by $ds^2=g_{11}du_1^2$. Then $ds=\sqrt{g_{11}}\,du_1$. Thus $\sqrt{g_{11}}$ is the ratio of distance advanced to increment in coördinate u_1. For example, if $ds=\frac{1}{2}$ for $du_1=1$, $\sqrt{g_{11}}=\frac{1}{2}$. This means that if a unit of length is placed on the coördinate line $u_2=$constant, one half of the unit extends from the line u_1 to u_1+1. A similar explanation is given for $\sqrt{g_{22}}$. If θ is the angle between the coördinate lines, $g_{12}=g_{21}=\cos\theta\sqrt{g_{11}}\sqrt{g_{22}}$. The metrical structure of the surface is known if we determine the g's for every point of the surface. Let us now consider the application of the methods of analytic geometry to physics.

5. A metric for a geometry for physics. Classical physics was founded on the assumption that physical space is Euclidean. This means that a set of equal rigid rods can be fitted together to form a cubical lattice of finite extent. The lines of the lattice may be used as the lines of a Cartesian coördinate system. Cartesian coördinates directly express the distance of a point from a coördinate plane and hence have direct physical significance. If Cartesian coördinates are symbolized by x_1, x_2, x_3, the metric of Euclidean space is expressed by the formula $ds^2=dx_1^2+dx_2^2+dx_3^2$. To the first approximation, at least, physical space is Euclidean, and this fact explains the universal application of Euclidean geometry in classical mechanics.

The special theory of relativity provided a basis for a four-dimensional space-time relational structure of events. In addition to three spatial coördinates x_1, x_2, x_3 there was introduced a fourth coördinate, the value of which is directly related to the time indicated by a clock. If t is time indicated by a clock, we may define $x_4=ict$. Then

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

expresses the metric of space-time. ds, the invariant interval between two events, is thus expressed in terms of differences of spatial coördinates and the time.

The general theory of relativity assumes that space-time is a continuum characterized by a Riemannian metric. In a gravitational field the positional relations of rigid bodies do not satisfy the propositions of Euclidean geometry. It is not possible to build finite Cartesian lattices out of equal rigid rods. The rate of clocks is affected by a gravitational field. Hence the metrical structure of a space-time region containing a gravitational field cannot be expressed by the formula for ds used in the special theory. The more general Riemannian formula

$$ds^2 = \sum g_{ik} dx_i dx_k$$

is necessary. The g_{ik} have a physical significance that may be defined by a procedure similar to the one for the two-dimensional surface.

The theory that physical space-time is Riemannian raises the problem of how the standard of measure for ds is set up at a particular space-time point. In ordinary space this is accomplished by bringing a standard of length to the point. But the interval of space-time contains spatial and temporal factors. The metrical evaluation of ds may be made with the aid of the special theory of relativity. In a relatively small space-time region it is possible to select a frame of reference relative to which there is no gravitational field. A gravitational field is relative to a frame of reference and will vanish relative to a suitable accelerated frame. For example, there is no gravitational field relative to an elevator which is falling freely towards the surface of the earth. Relative to the frame with respect to which there is no field and in which the coördinates of an event are x_1, x_2, x_3, x_4, the interval between two events may be expressed by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2;$$

and dx_1, dx_2, dx_3, dx_4 may be determined by rigid rods and clocks as in special relativity, and hence the value of the corresponding ds can be calculated.

The geometrical significance of the g_{ik} is part of their physical significance. The g_{ik} also have a dynamical significance, for they are the potentials of the gravitational field. The law of gravitation expresses a condition on the g's and their derivatives. The fundamental law of motion is that a free particle describes a geodesic in curved space-time. In this sense physics is reduced to geometry, but geometry is a branch of physics.

6. Summary. This paper may be summarized by a restatement of the relation between physical geometry and abstract geometry. Typical propositions of Euclidean geometry may be formulated as generalizations from experiences of practically rigid bodies. Such laws are expressed in terms of quantities which may be determined within limits of precision. The next step is to assume that the propositions hold exactly for a set of objects, such as ideal rigid bodies. Propositions with a precisely defined content may be reduced to a set of axioms

from which theorems can be deduced. The status in reality of ideal objects is uncertain. Historically the attempt has been made to give them reality in a transcendent realm or to view them as constructions in pure intuition. The problem of the ontological status of the objects of geometry is avoided by eliminating the empirical reference of the concepts. The axioms then become postulates which implicitly define the formal properties of the objects of the concepts. Thus generalizations from experience become transformed into definitions. The self-evidence which has been attributed to the axioms of Euclidean geometry is founded on their status as definitions. The proposition that a straight line is the shortest distance between two points is self-evident in the sense that it may be used as the definition of a straight line.

Once we have the concept of abstract geometry, it is possible to create new abstract geometries and then seek physical interpretations of them. The interest in differential geometry stimulated by the general theory of relativity has resulted in the invention of non-Riemannian geometries. The geometry of Weyl, for example, is based upon the assumption that the standard of distance is a function of position. But such developments are beyond the scope of this paper.

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UPPER LIMITS TO THE REAL ROOTS OF A REAL ALGEBRAIC EQUATION

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1. Introduction. There are in the literature a number of theorems concerning upper limits to the real roots of an algebraic equation with real coefficients. The more popular of these theorems are those which can be easily remembered and are easy to apply. In many cases, however, they give very high limits.

We shall give three theorems, the first due to Lagrange, the second due to Jean J. Bret, and the third to Newton. For proofs of these theorems the reader may refer to Dickson's *First Course in the Theory of Equations* for the first two theorems, and to Burnside and Panton's *Theory of Equations* for the third.

THEOREM I. *If in a real equation (an equation with real coefficients)*

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad (a_0 > 0),$$

the first negative coefficient is preceded by k coefficients which are positive or zero,

and if G denotes the greatest of the numerical values of the negative coefficients, then each real root is less than $1+\sqrt[k]{G/a_0}$.

THEOREM II. *If, in a real algebraic equation in which the coefficient of the highest power of the unknown is positive, the numerical value of each negative coefficient be divided by the sum of all the positive coefficients which precede it, the greatest quotient so obtained increased by unity is an upper limit to the roots.*

THEOREM III. *If the real function*

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

and its first n derivatives evaluated for x equal to a positive number h are all positive or zero, then h is an upper limit to the real roots of the equation $f(x)=0$.

It is the object of this paper to present two other theorems, believed to be new, which give lower limits for the roots than are given by Theorems I and II.

2. Two theorems on limits of roots.

THEOREM IV. *If, in an equation with real coefficients,*

$$\begin{aligned}(1)\qquad f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{k-2}x^{n-(k-2)} + a_{k-1}x^{n-(k-1)} + a_kx^{n-k} \\ + a_{k+1}x^{n-(k+1)} + \cdots + a_{n-1}x + a_n = 0, \qquad (a_0 > 0),\end{aligned}$$

the first negative coefficient is preceded by k coefficients which are positive or zero, and if G denotes the greatest of the numerical values of the negative coefficients, then each real root greater than h is less than L_1, where

$$(2)\qquad L_1 = 1 + \sqrt[k]{(G + a_0(h-1)^k - (h-1)\sum_{s=0}^{k-1} a_sh^{(k-1)-s})/a_0},$$

and where h satisfies the inequality

$$(3)\qquad G - (h-1)\sum_{s=0}^{k-1} a_sh^{(k-1)-s} > 0.$$

THEOREM V. *With the same hypotheses as in Theorem IV, for $x>2$ an upper limit to the real roots of $f(x)=0$ is L_2, where*

$$(4)\qquad L_2 = 1 + \sqrt[k]{(G - \sum_{s=0}^{k-1} a_s)/a_0},$$

whenever the expression under the radical is greater than one.

3. Proof of Theorems IV and V. For x positive, $f(x)$ will remain unchanged or be reduced in value if we replace each coefficient after a_{k-1} by $-G$. Hence

$$f(x) \geqq a_0x^n + a_1x^{n-1} + \cdots + a_{k-2}x^{n-(k-2)} + a_{k-1}x^{n-(k-1)} - G(x^{n-k} + \cdots + 1).$$

If $x \neq 1$, then

$$f(x) \geqq a_0x^n + a_1x^{n-1} + \cdots + a_{k-2}x^{n-(k-2)} + a_{k-1}x^{n-(k-1)} - G\,\frac{x^{n-(k-1)} - 1}{x - 1}\cdot$$

Then, dropping positive G from the numerator, we have

$$f(x) > \frac{x^{n-(k-1)}\left[(a_0x^{k-1} + a_1x^{k-2} + \cdots + a_{k-2}x + a_{k-1})(x - 1) - G\right]}{(x - 1)}\cdot$$

For $x>1$, $f(x)$ will be greater than zero, and hence x will not be a root if

$$(5)\qquad (a_0x^{k-1} + a_1x^{k-2} + \cdots + a_{k-2}x + a_{k-1})(x - 1) - G > 0.$$

From the identity

$$x^{k-1} \equiv x^{k-1} - (x - 1)^{k-1} + (x - 1)^{k-1}$$

it follows, for $x \geqq h \geqq 1$, that

$$(6)\qquad x^{k-1} \geqq h^{k-1} - (h - 1)^{k-1} + (x - 1)^{k-1};$$

and hence (5) will be verified if the following inequality holds:

$$(7)\qquad \begin{aligned}(a_0h^{k-1} - a_0(h - 1)^{k-1} + a_0(x - 1)^{k-1} + a_1x^{k-2} + \cdots\\ + a_{k-2}x + a_{k-1})(x - 1) - G > 0.\end{aligned}$$

For $x>h$, this will be true if we have

$$(8)\qquad \begin{aligned}[(a_0h^{k-1} + a_1h^{k-2} + \cdots + a_{k-2}h + a_{k-1}) - a_0(h - 1)^{k-1}](h - 1)\\ + a_0(x - 1)^k - G > 0.\end{aligned}$$

Thus, if inequality (8) holds, then inequality (5) holds, and finally $f(x)$ will be greater than zero. Hence an x for which inequality (8) holds cannot be a real root.

Taking $h=1$ in the inequality (8) gives Theorem I.

From inequality (8) it also follows that $x<L_1$, where L_1 is defined in (2). Since, in deriving inequality (6), x was assumed to satisfy $x \geqq h \geqq 1$, h must be chosen so that (3) is satisfied. Thus Theorem IV is established.

To prove Theorem V, put $h=1$ in the bracket of inequality (8), and $h=2$ in its multiplier, $h-1$. This gives $x<L_2$, where L_2 is defined in (4), provided $x>2$. The expression under the radical must be greater than one. This completes the proof of Theorem V.

Since inequality (3) is the reverse of (5), with a change of variable, it would appear at first glance that little has been accomplished. However, if the numbers $a_0, a_1, \cdots, a_{k-1}$, are such that a considerable difference exists between

$$a_0(h - 1)^k \qquad \text{and} \qquad (h - 1)\sum_{s=0}^{k-1} a_s h^{(k-1)-s},$$

then an h may be readily chosen which will materially decrease the upper limit given by Theorem IV as compared with the limit given by Theorem I.

For example, an upper limit to the roots of the equation

$$x^5 + 10x^4 - 61x^3 + 1 = 0$$

is, by Theorem I,

$$1 + \sqrt[2]{\frac{61}{1}} = 1 + 7.8 = 8.8.$$

By Theorem IV, an upper limit is

$$1 + \sqrt[2]{\frac{61 + (h-1)^2 - (h-1)(h+10)}{1}}.$$

For $h=5$, $61>(h-1)(h+10)$; hence an upper limit is

$$1 + \sqrt[2]{1 + 4^2} = 1 + 4.1 = 5.1.$$

4. Improvements on the limits given by Theorems II and III. When $k=1$, Theorems I and II often give very poor limits for roots. Thus, for the equation

$$x^5 - x^4 - x^3 - 2x^2 - 3x - 256 = 0,$$

Theorem I gives for an upper limit

$$1 + \sqrt[1]{256} = 257,$$

and Theorem II gives

$$1 + \frac{256}{1} = 257.$$

To apply Theorem II more effectively put $x=my$ and apply Theorem II to the resulting equation in y. For this example we get the upper limit

$$x = m\left(1 + \frac{256}{m^5}\right) = m + \frac{256}{m^4}.$$

We may in general choose by inspection an m which will give a satisfactory upper limit. In this particular case we may apply the theory of maxima and minima to obtain a low limit; thus

$$\frac{dx}{dm} = 1 - \frac{1024}{m^5} = 0,$$

from which, $m=4$. Hence we obtain the limit

$$x = 4 + \frac{256}{256} = 5.$$

Finally, to improve Theorem III, when $k=1$ or $k=2$, and a_1 or a_2 is large and negative, we proceed from the well-known rule that the remainders obtained by

dividing $f(x)$ by $(x-h)$, the quotient thus obtained by $(x-h)$, and so on, are respectively $f'(h)/1, \cdots, f^n(h)/n!$ that is, they are the coefficients of $(x-h)^0$, $(x-h)', \cdots, (x-h)^n$ when $f(x)$ is expanded in a Taylor Series in powers of $(x-h)$. Hence by Theorem III if these remainders are all positive or zero, then h is an upper limit to the real roots of $f(x)=0$.

If a_1 is negative* then, if a_0 is positive, we try $h \geqq -a_1/n$, since, in repeated synthetic division by $x-h$, this will make the lowest entry in the second column (which equals $f^{(n-1)}(h)/(n-1)!$) positive or zero.

Thus for the equation

$$x^3 - 20x^2 + 164x - 400 = 0$$

we should choose $h \geqq 20/3$ or $h=7$. When repeated synthetic division by $x-7$ is carried out, we obtain the successive remainders 111, 31, 1, 1. Hence 7 is an upper limit to the roots. Note that Theorems I and II give 21 as an upper limit.

According to the remark made above, if all the remainders obtained by dividing $f(x)$ and the successive quotients by $x-h$, where $h \geqq -a_1/n$, are not positive, then certainly in the new equation

$$\frac{f^n(h)}{n!}(x-h)^n + \frac{f^{n-1}(h)}{(n-1)!}(x-h)^{n-1} + \cdots + \frac{f'(h)}{1}(x-h) + f(h) = 0 \tag{9}$$

it will be true that $k \geqq 2$.

Theorems II and IV applied to this equation may give relatively better limits than when applied to $f(x)=0$. Since the roots of the original equation have been diminished by h, the upper limit to the roots of $f(x)=0$ will be h plus the upper limit to the roots of equation (9).

ON SOME GENERALIZATIONS OF CAUCHY'S CONDENSATION AND INTEGRAL TESTS

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Cauchy's celebrated condensation test (C-Test) in its most general form runs as follows:—

C-TEST. *Let $\phi(x)$ be a positive, single-valued function of x which steadily decreases to zero as x increases to infinity and $F(x) \equiv a^x\phi(a^x)$, a being a real number greater than unity; then the two series $\sum_1^\infty \phi(n)$ and $\sum_1^\infty F(n)$ converge and diverge together.*

The proofs of the test which are usually given hold for integral values of a. Chrystal [1] gives an incomplete proof of the general case. Borel [2] and Brom-

* If a_1 is positive, but a_2 is negative, then h should be taken to satisfy the relation

$$h \geqq \sqrt[2]{\frac{(a_2)\,(2!)\,(n-2!)}{a_0(n!)}}.$$

wich [**3**] dismiss it with the remark that the proof can be easily extended by taking the integral part of a^n when $a>1$. The test in its general form was first proved by Kohn [**4**] and afterwards by Hill [**5**]. Recently Krishnaswamy Ayyangar [**6**] and Moritz [**7**] have given two independent proofs of this test. Yet another can be added in the lines of the proof of the test below.

Cauchy's Integral test may be stated as follows:—

I-TEST. *Under the conditions imposed on $\phi(x)$ in the C-Test, the series $\sum^\infty \phi(n)$ and the integral $\int^\infty \phi(x)dx$ converge and diverge together.*

A generalization of the C-Test is also known (see, *e.g.*, Knopp [**8**] and Fort [**9**], which in a more extended form can be stated as follows:—

If $\phi(x)$ satisfies the restrictions of the C-Test, and g_n is a positive increasing sequence tending to infinity with n, then

(i) *$\sum_1^\infty \phi(n)$ converges if $\sum_1^\infty (g_{n+1}-g_n)\phi(g_n)$ converges,*

(ii) *$\sum_1^\infty \phi(n)$ diverges if $\sum_1^\infty (g_{n+1}-g_n)\phi(g_{n+1})$ diverges,*

(iii) *$\sum_1^\infty \phi(n)$, $\sum_1^\infty (g_{n+1}-g_n)\phi(g_n)$, $\sum_1^\infty (g_{n+1}-g_n)\phi(g_{n+1})$ converge and diverge together if the sequence $(g_{n+1}-g_n)$ is bounded, or if for every positive integer k, $g_{k+1}-g_k<\lambda(g_k-g_{k-1})$, where λ is constant.*

Proof: Since $\phi(g_k)>\phi(x)>\phi(g_{k+1})$ for $g_{k+1}>x>g_k$ we have

$$(g_{k+1}-g_k)\phi(g_k) \geqq \int_{g_k}^{g_{k+1}} \phi(x)dx \geqq (g_{k+1}-g_k)\phi(g_{k+1}). \tag{1}$$

Adding these inequalities for $k=1, 2, \cdots, n$, we get

$$\sum_1^n (g_{k+1}-g_k)\phi(g_k) \geqq \int_{g_1}^{g_{n+1}} \phi(x)dx \geqq \sum_1^n (g_{k+1}-g_k)\phi(g_{k+1}).$$

From this, with the help of the I-test, follow the results (i) and (ii).

To prove (iii), suppose $g_n = g_{n-1} < \mu$ for every n, where μ is constant; then

$$\begin{aligned} 0 &< \sum_2^n (g_k - g_{k-1})\phi(g_{k-1}) - \sum_2^n (g_k - g_{k-1})\phi(g_k) \\ &\leqq \mu \sum_2^n \{\phi(g_{k-1}) - \phi(g_k)\} \\ &= \mu\{\phi(g_1) - \phi(g_n)\} < \mu\phi(g_1). \end{aligned}$$

Thus the two series $\sum^\infty (g_n-g_{n-1})\phi(g_n)$ and $\sum^\infty (g_n-g_{n-1})\phi(g_{n-1})$ converge and diverge together. The result with the I-test proves the first part of (iii).

To prove the second part of (iii), we note that if for every k,

$$g_{k+1} - g_k \leqq \lambda(g_k - g_{k-1}),$$

the inequality (1) can be replaced by either of the following:

$$\lambda(g_k - g_{k-1})\phi(g_k) \geqq \int_{g_k}^{g_{k+1}} \phi(x)dx \geqq (g_{k+1} - g_k)\phi(g_{k+1}),$$

$$(g_{k+1} - g_k)\phi(g_k) \geqq \int_{g_k}^{g_{k+1}} \phi(x)dx \geqq \frac{1}{\lambda}(g_{k+2} - g_{k+1})\phi(g_{k+1}).$$

Adding these inequalities for $k=2, 3, \cdots, n$ and $k=1, 2, \cdots, n$, respectively, we get

$$\lambda\sum_{2}^{n}(g_k - g_{k-1})\phi(g_k) \geqq \int_{g_2}^{g_{n+1}} \phi(x)dx \geqq \sum_{3}^{n+1}(g_k - g_{k-1})\phi(g_k), \tag{2}$$

$$\sum_{1}^{n}(g_{k+1} - g_k)\phi(g_k) \geqq \int_{g_1}^{g_{n+1}} \phi(x)dx \geqq \frac{1}{\lambda}\sum_{2}^{n+1}(g_{k+1} - g_k)\phi(g_k). \tag{3}$$

These inequalities taken with the I-Test prove the second part of (iii).

Let $g(n)=a^n$ where $a>1$, the inequality (1) can be written as

$$(a - 1)a^k\phi(a^k) \geqq \int_{a^k}^{a^{k+1}} \phi(x)dx \geqq \frac{a-1}{a} a^{k+1}\phi(a^{k+1}); \tag{4}$$

whence by adding these inequalities for $k=1, 2, \cdots$, we get Cauchy's C-Test.

A generalization of the I-Test will now be deduced. Transform the integral in (4) by the substitution $x=a^t$, and obtain

$$(a - 1)a^k\phi(a^k) \geqq \int_{k}^{k+1} a^t\phi(a^t)\log a dt \geqq \frac{a-1}{a} a^{k+1}\phi(a^{k+1}).$$

The addition of these inequalities for $k=1, 2, \cdots$, proves the following generalization of the I-Test.

If $\phi(x)$ satisfies the restrictions set forth in the C-Test and $F(x)=a^x\phi(a^x)$, then $\sum^\infty F(n)$ and $\int^\infty F(x)dx$ converge and diverge together.

It may be pointed out that $F(x)$ does not necessarily satisfy the restrictions of the I-Test. The function $F(x)$ is not necessarily a decreasing or even monotonic function. Thus Cauchy's Integral test holds under less stringent conditions than are usually inposed on it.

Similarly from (2) and (3) the following result can be deduced:

If (i) $\phi(x)$ satisfies the restrictions set forth in the C-Test, (ii) $g(n)$ is a positive increasing sequence tending to infinity with n, (iii) the sequence $(g_n - g_{n-1})$ is either bounded or $(g_{n-1} - g_n) \leqq \lambda(g_n - g_{n-1})$ for every n, where λ is constant, (iv) $h(x)$ is a monotonic increasing function of x defined in the interval $(1, \infty)$ such that $h(n) = g(n)$ for all values of n and $h'(x)$ exists, then

$$\sum^{\infty}(g_{n+1} - g_n)\phi(g_n), \quad \sum^{\infty}(g_{n+1} - g_n)\phi(g_{n+1}), \quad \int^{\infty}\phi(h(x))h'(x)dx$$

converge and diverge together.

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ON LOCI ASSOCIATED WITH OSCULANTS AND PENOSCULANTS OF A PLANE CURVE

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1. A general principle. It is an elementary fact, well known for a long time, that the center of the circle of curvature of a plane curve lies on the envelope of the normal to the curve. Cesàro* showed that the center of the osculating conic lies on the envelope of the axis of aberrancy. It can be shown directly, by the method used in §2, that the focus of the osculating parabola lies on the envelope of the circle whose diameter is that half of the radius of curvature terminating at the point of contact. For the purpose of this paper, it is significant that the normal is the locus of centers of penosculating circles, the axis of aberrancy is the locus of centers of penosculating conics, and the circle mentioned in the previous sentence is the locus of foci of penosculating parabolas. A statement and proof of the general principle which the above examples illustrate follows.

Let C be a plane curve possessing values for $\rho, \rho', \cdots, \rho^{(n-3)}$ at all of its points, where ρ indicates the radius of curvature of C, and the superscripts indicate differentiation with respect to arc length s measured along C. C will be referred to as the base curve. Let ϕ denote a class of curves whose cartesian equation contains n independent parameters. Suppose there exists at each point M of C a unique member of ϕ, denoted by Γ, which has n-point contact with C there, and a one parameter family g, consisting of those members of ϕ which have $(n-1)$-point contact with C there. (Γ is called the osculating member of ϕ and g the family of penosculating members of ϕ.) Let P be a point associated with Γ, whose coördinates X and Y with respect to the reference frame consisting of the tangent and normal to Γ at any point are functions of $\bar{\rho}, \bar{\rho}', \cdots, \bar{\rho}^{(n-3)}$, where these symbols represent the radius of curvature of Γ and its successive derivatives as to arc length $\bar{s}$ on Γ, all of which symbols are functions of $\bar{s}$. The existence of the quantities just mentioned is postulated. Let points corresponding to P for all members of g be defined as those points whose coördinates are the same functions of $\bar{\rho}_i, \bar{\rho}_i', \cdots, \bar{\rho}_i^{(n-3)}$ as X and Y are of $\bar{\rho}, \bar{\rho}', \cdots, \bar{\rho}^{(n-3)}$,

* Nouvelles Annales de Mathématiques, vol. 9, 1890, p. 150.

where the $\bar{\rho}$'s with subscript i are formed for the ith member of g, and the reference frame consists of the tangent and normal to this ith member of g. Suppose the points corresponding to P form a locus L, whose cartesian equation referred to the tangent and normal to C at M is analytic, of the form

$$F(x, y, \rho, \rho', \cdots, \rho^{(n-4)}) = 0.$$

Then as M moves along C, P will ordinarily trace out a curve associated with C, and L will determine an envelope. Under the conditions stated, we have the following theorem:

THEOREM: *P will lie on the envelope of L.*

Proof: To show that P lies on the envelope of L, we must show that X and Y satisfy both $F=0$ and

$$\frac{\delta F}{ds} \equiv \frac{\partial F}{\partial x}\left(\frac{y}{\rho} - 1\right) + \frac{\partial F}{\partial y}\left(-\frac{x}{\rho}\right) + \frac{\partial F}{\partial s} = 0$$

at a typical point M on C, the reference frame being the tangent and normal common to C and Γ at M. At M, due to the fact that Γ osculates C,

$$\bar{\rho} = \rho, \bar{\rho}' = \rho', \cdots, \bar{\rho}^{(n-3)} = \rho^{(n-3)}. \tag{1}$$

Furthermore, let us take M as the origin for measuring both s and $\bar{s}$. If Γ were taken as a base curve, the equation of L would be

$$\bar{F} \equiv F(x, y, \bar{\rho}, \bar{\rho}', \cdots, \bar{\rho}^{(n-4)}) = 0.$$

Since X and Y satisfy this equation for every value of $\bar{s}$, we see that

$$F(X, Y, \bar{\rho}, \bar{\rho}', \cdots, \bar{\rho}^{(n-4)}) = 0$$

is an identity in $\bar{s}$. Hence the derivative of the left member with respect to $\bar{s}$ will be zero for all values of $\bar{s}$. Therefore, P being a fixed point with respect to a changing $\bar{s}$ so that

$$\frac{dX}{d\bar{s}} = \frac{Y}{\bar{\rho}} - 1 \quad \text{and} \quad \frac{dY}{d\bar{s}} = -\frac{X}{\bar{\rho}},$$

we have

$$\frac{\partial \bar{F}}{\partial X}\left(\frac{Y}{\bar{\rho}} - 1\right) + \frac{\partial \bar{F}}{\partial Y}\left(-\frac{X}{\bar{\rho}}\right) + \frac{\partial \bar{F}}{\partial \bar{s}} = 0.$$

But at M, using (1) and the fact that $\bar{s}=s$, this last equation reduces to

$$\left[\frac{\partial F}{\partial x}\left(\frac{y}{\rho} - 1\right) + \frac{\partial F}{\partial y}\left(-\frac{x}{\rho}\right) + \frac{\partial F}{\partial s}\right]_{\substack{x=X\\y=Y}} = 0.$$

Hence at M, the coördinates of P satisfy both $F=0$ and $\dfrac{\delta F}{ds}=0$. But M is any point of C. Hence the theorem is proved.

2. Illustrations. The direct investigation of envelopes arising in two families of penosculants will show the significance of this principle in the theory of osculants and penosculants of a plane curve. Throughout, the coördinate reference frame will consist of the tangent and normal to the base curve at the point of contact.

The locus of the centers of the members of the family of penosculating equilateral hyperbolas defined at a point of a plane curve is a circle, tangent to the curve on its convex side, with a diameter equal to the radius of curvature of the base curve.* The equation of this circle is

$$x^2 + y^2 + \rho y = 0. \tag{2}$$

To get its envelope as the point of contact moves along the base curve, differentiate (2) as to s and use the relations

$$\frac{dx}{ds} = \frac{y}{\rho} - 1 \quad \text{and} \quad \frac{dy}{ds} = -\frac{x}{\rho}\cdot$$

This leads to $3x - \rho' y = 0$. Solving this equation simultaneously with (2), we get $x=0$, $y=0$ and

$$x = -\frac{3\rho\rho'}{\rho'^2 + 9} \qquad y = -\frac{9\rho}{\rho'^2 + 9}\cdot$$

The first pair of values correspond to the base curve, and the second to the center of the osculating equilateral hyperbola. Thus, the envelope consists of the base curve and *the locus of the center of the osculating equilateral hyperbola*, illustrating the principle stated in §1.

The locus of the vertices of penosculating parabolas defined at a point of a plane curve is a tacnodal, circular quartic whose equation is

$$2(x^2 + y^2)^2 - 5\rho x^2 y + 4\rho y^3 + 2\rho^2 y^2 = 0.$$

A parametric representation—more convenient for our purposes—is

$$x = -\frac{\rho t(2 + t^2)}{2(1 + t^2)^2}, \qquad y = \frac{\rho t^2}{2(1 + t^2)^2}\cdot \tag{3}$$

To get the envelope of this quartic, we employ the following method. Compute the indicated derivatives from (3), and substitute in

$$\begin{vmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} - \dfrac{y}{\rho} + 1 \\[2ex] \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial s} + \dfrac{x}{\rho} \end{vmatrix} = 0. \tag{4}$$

This leads to the equation

$$3t^7 - \rho' t^6 - 3t^5 + \rho' t^4 - 6t^3 + 2\rho' t^2 = 0.$$

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* J. W. Lasley, Jr., Penosculating Conics of a Plane Curve, Bulletin of the American Mathematical Society, vol. 37, 1931, p. 77.

The roots of this equation are 0, 0, $\pm i$, $\pm\sqrt{2}$ and $\rho'/3$. These values when substituted for t in (3) give the parametric equations of the various branches of the envelope of the quartic. The value $\rho'/3$ gives

$$x = -\frac{3\rho\rho'(\rho'^2 + 18)}{2(\rho'^2 + 9)^2}, \qquad y = \frac{9\rho\rho'^2}{2(\rho'^2 + 9)^2}.$$

But these are parametric equations of *the locus of the vertex of the osculating parabola.**

3. A dual principle. The general principle given in §1 can easily be dualized by a few appropriate changes in wording. The meaning of this dual will be clear from the following investigation of some geometry concerning families of penosculating equilateral hyperbolas, penosculating parabolas, and penosculating conics.

The asymptotes of the family of penosculating equilateral hyperbolas envelop a hypocycloid of three cusps formed by a circle of radius $\rho/2$ rolling in a circle of radius $3\rho/2$. A parametric representation for the hypocycloid is

(5) $$x = 2\rho \sin 2\alpha \sin^2 \alpha, \qquad y = -2\rho \cos 2\alpha \cos^2 \alpha.$$

Applying to (5) the technique used in investigating the envelope of the locus of vertices of penosculating parabolas, we get as roots of the equation obtained from (4) 0°, 60°, 120°, 90°, and $\frac{1}{2}$ arc cot $(-\rho'/3)$. The first three correspond to the cusps, and hence must not be used in getting the envelope; the fourth value corresponds to the origin; and the last value gives

$$x = \frac{3\rho(\pm\sqrt{\rho'^2 + 9} + \rho')}{\rho'^2 + 9}, \qquad y = \frac{\rho\rho'(\pm\sqrt{\rho'^2 + 9} - \rho')}{\rho'^2 + 9}.$$

These can be shown to be the parametric equations of the envelopes of the asymptotes of the osculating equilateral hyperbola. That is, *a part of the envelope of* (5) *consists of the envelopes of the asymptotes of the osculating equilateral hyperbola.* The previous statement embodies an example of the dualized form of the principle of §1.

The tangents at the vertices of penosculating parabolas envelope a hypocycloid of three cusps formed by a circle of radius $\rho/8$ rolling in a circle of radius $3\rho/8$. Applying the method just used, we obtain as the parametric equations of one branch of the envelope of this hypocycloid

$$x = -\frac{27\rho\rho'}{(\rho'^2 + 9)^2}, \qquad y = \frac{\rho\rho'^2(9 - \rho'^2)}{2(\rho'^2 + 9)^2}.$$

This curve can be shown to be *the envelope of the tangent at the vertex of the osculating parabola.*

As a final illustration of the dual principle, the envelope of the axes of penosculating conics defined at a point of a curve is a parabola, known as Tran-

* A. Colucci, Sulle coniche osculatrici ad una data curva, Giornale di Matematiche di Battaglini, vol. 71, 1933, p. 166.

son's parabola, which has some interesting relations to the osculants considered in this paper.* An application of the principle shows that as the point of contact moves along the base curve, *the axes of the osculating conic are always tangent to the envelope of Transon's parabola.*

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A NEW PAIR OF AMICABLE NUMBERS

B. H. BROWN, Dartmouth College

There are 68 known pairs of amicable numbers, of which 41 are (both) even, and 27 (both) odd.† With three exceptions these pairs have the property that the smallest prime factor occurs to the same power in each of the numbers of the pair. The exceptions are the two pairs

$$2^3\cdot 19\cdot 41,\quad 2^5\cdot 199 \qquad \text{and} \qquad 2^3\cdot 41\cdot 467,\quad 2^5\cdot 19\cdot 233,$$

both due to Euler, and the pair

$$2^5\cdot 37,\quad 2\cdot 5\cdot 11^2,$$

due to Paganini.

In an effort to produce an odd pair of exceptional type, many simple cases were investigated, among which was

$$3^3\cdot 5\cdot 7\cdot p,\quad 3^m\cdot 5\cdot 7\cdot q,$$

where $m<3$, and p and q are primes. This leads immediately to the necessary and sufficient conditions $m=1$, $p=13$, $q=139$. Hence

$$12285 = 3^3\cdot 5\cdot 7\cdot 13,\quad 14595 = 3\cdot 5\cdot 7\cdot 139$$

are amicable.

I have further verified that of all amicable number pairs, the smaller number of which is less than 15,000, the Euler list is complete, except for the Paganini pair, and the pair added in this paper.

A HISTORICALLY INTERESTING FORMULA FOR THE AREA OF A QUADRILATERAL

J. L. COOLIDGE, Harvard University

The amount of information available in the literature of mathematics bearing on the quadrilateral, the general quadrilateral, the cyclic quadrilateral, the

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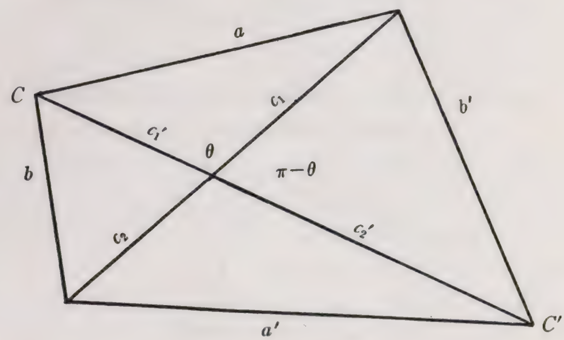
* See E. J. Wilczynski, Some Remarks on the Historical Development and the Future Prospects of the Differential Geometry of Plane Curves, Bulletin of the American Mathematical Society, vol. 22, 1916, p. 323, and J. W. Lasley, Jr., *loc. cit.*, p. 79.

† Dickson: *History of the Theory of Numbers*, 1919, vol. 1, chapter 1. A pair of numbers is said to be amicable if each is equal to the sum of the proper factors of the other.

complete quadrilateral, *etc.*, is discouragingly large. Yet there seems to be no part of the science so far from exhaustion as elementary geometry. It has seemed to me that there must be connecting links between different known formulas which were worth investigating, not only for their own sakes but also for historical reasons. Here is one which, so far as I can find out, is new, and which seems to me to come into that category.

Let the pairs of opposite sides of a quadrilateral be aa', bb' while the diagonals are cc'. The first diagonal shall be divided by the intersection into the parts



c_1c_2 while the second is divided into the parts $c_1'\,c_2'$. Let one angle of the diagonals be θ, let the area be A, and finally let

$$2s = a + a' + b + b'.$$

Then

$$4A^2 = c^2c'^2 \sin^2\theta,$$

and

$$a^2 = c_1^2 + c_1'^2 - 2c_1c_1' \cos\theta,$$

$$\frac{c_1c_1'}{2} = \frac{c_1^2 + c_1'^2 - a^2}{4\cos\theta}.$$

Similarly,

$$\frac{c_2c_1'}{2} = -\frac{c_2^2 + c_1'^2 - b^2}{4\cos\theta},$$

$$\frac{c_2c_2'}{2} = \frac{c_2^2 + c_2'^2 - a'^2}{4\cos\theta},$$

$$\frac{c_1c_2'}{2} = -\frac{c_1^2 + c_2'^2 - b'^2}{4\cos\theta}.$$

Adding, we obtain

$$\frac{cc'}{2} = \frac{(b^2 + b'^2) - (a^2 + a'^2)}{4\cos\theta}.$$

Hence

$$A^2 = \frac{c^2c'^2}{4}\left[1 - \cos^2\theta\right]$$

$$= \frac{c^2c'^2}{4} - \frac{\left[(b^2 + b'^2) - (a^2 + a'^2)\right]^2}{16}$$

$$= -\frac{(a^4 + a'^4 + b^4 + b'^4) + 2(a^2 + a'^2)(b^2 + b'^2) - 2(a^2a'^2 + b^2b'^2)}{16} + \frac{c^2c'^2}{4}$$

$$= (s-a)(s-a')(s-b)(s-b') - \tfrac{1}{4}\left[a^2a'^2 + b^2b'^2 + 2aa'bb' - c^2c'^2\right] \cdot$$

$$A^2 = (s-a)(s-a')(s-b)(s-b') - \tfrac{1}{4}\left[aa' + bb' + cc'\right]\left[aa' + bb' - cc'\right].$$

Now a very famous theorem due to Ptolemy (*circa* 139 A.D.) tells us that a necessary and sufficient condition that the vertices of a quadrilateral should lie on a circle is that

$$aa' + bb' = cc'.$$

But in that case we have for the area of a cyclic quadrilateral

$$A = \sqrt{(s-a)(s-a')(s-b)(s-b')},$$

an almost equally famous formula due to Brahmagupta (*circa* 728 A.D.)

I should, perhaps, mention in this connection that in 1842 there appeared in Grunert, *Beiträge zur reinen und angewandten Mathematik*, vol. 2, two proofs by Bretschneider and Strehlke, both rather clumsy I think, of the formula

$$A^2 = (s-a)(s-a')(s-b)(s-b') - aa'bb' \cos^2 \frac{C + C'}{2} \cdot$$

ON THE GEOMETRY OF THE TRIANGLE

J. R. MUSSELMAN, Western Reserve University

In this note I wish to point out another application of some ideas presented in my article on The Line of Images (this MONTHLY, 1938, p. 421.) Using the same notation, let T be any point on the circumcircle of triangle $A_1A_2A_3$. The line through T parallel to its line of images is

(1) $$Tx - \sigma_3\bar{x} = T^2 - \sigma_3/T.$$

This line cuts the side A_2A_3 in the point whose coördinate is

$$Z_1 = (T^2 - \sigma_3/T + t_1t_2 + t_1t_3)/(t_1 + T).$$

The line through T', the diametrically opposite point of T, parallel to its line of images is

(2) $$Tx + \sigma_3\bar{x} = -T^2 - \sigma_3/T.$$

This line cuts A_2A_3 in the point whose coördinate is

$$Z_2 = (T^2 + \sigma_3/T + t_1t_2 + t_1t_3)/(t_1 - T).$$

The midpoint of Z_1Z_2 has the coördinate

$$Z = \frac{(\sigma_2 + T^2)t_1}{t_1^2 - T^2}. \tag{3}$$

The coördinates of the midpoints of the segments cut on the sides A_3A_1 and A_1A_2 by the lines (1) and (2) may be written down from (3) by replacing t_1 by t_2 and t_3 respectively. These three points lie on a line whose equation is

$$(\sigma_3 + \sigma_1T^2)x + \sigma_3(\sigma_2 + T^2)\bar{x} = 0. \tag{4}$$

The isogonal conjugate of the diameter TOT' is the equilateral hyperbola

$$T^2x^2 + (\sigma_3 - \sigma_1T^2)x - \sigma_3^2\bar{x}^2 + (\sigma_2\sigma_3 - \sigma_3T^2)\bar{x} + \sigma_2T^2 - \sigma_1\sigma_3 = 0.$$

The tangent at the orthocenter H of $A_1A_2A_3$ to the hyperbola is

$$(\sigma_3 + \sigma_1T^2)x - \sigma_3(\sigma_2 + T^2)\bar{x} = (\sigma_1^2 - \sigma_2)T^2 + \sigma_1\sigma_3 - \sigma_2^2.$$

Hence we have the theorem *that lines through the end points of any diameter TOT' of the circumcircle of $A_1A_2A_3$ parallel to their respective lines of images will cut the sides of $A_1A_2A_3$ in segments whose midpoints lie on the line, passing through the circumcenter of $A_1A_2A_3$, which is perpendicular to the tangent, at the orthocenter of $A_1A_2A_3$, to the hyperbola isogonally conjugate to the diameter TOT'.*

The intersection of the lines (1) and (2) is the point

$$Z = -\sigma_3/T^2. \tag{5}$$

Hence *the lines through the endpoints of any diameter TOT' of the circumcircle of $A_1A_2A_3$ parallel to their respective lines of images intersect on the circumcircle at that point which is the fourth intersection with the hyperbola isogonally conjugate to TOT'.*

The symmetric of (3) as to the midpoint of A_2A_3 is

$$d_1 = \frac{\sigma_3 + \sigma_1T^2}{T^2 - t_1^2}.$$

Hence *the reciprocal transversal of the line* (4) *is the line Δ of the Droz-Farny Theorem.*

NOTES ON CURVILINEAR MOTION IN THE PLANE

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Let $x=x(t)$, $y=y(t)$ describe the motion of a particle in the plane; let v_x, v_y, v, α_x, α_y, α, α_t, α_n have their conventional meanings; let v_ρ and α_ρ be the radial components of v and α respectively; let v_θ and α_θ be the transverse components of v and α respectively; let $C(x_c, y_c)$ be the center of curvature for the curve at any point $P(x, y)$, and let R be the radius of curvature of the curve for any point

$P(x, y)$. The following formulas are easy to derive, easy to remember, and easy to apply:

$$\alpha_t = \frac{v_x\alpha_x + v_y\alpha_y}{v}, \qquad \alpha_n = \frac{\begin{vmatrix} v_x & v_y \\ \alpha_x & \alpha_y \end{vmatrix}}{v},$$

$$v_\rho = \frac{xv_x + yv_y}{\rho}, \qquad v_\theta = \frac{\begin{vmatrix} x & y \\ v_x & v_y \end{vmatrix}}{\rho}, \qquad (\rho = \sqrt{x^2 + y^2})$$

$$\alpha_\rho = \frac{x\alpha_x + y\alpha_y}{\rho}, \qquad \alpha_\theta = \frac{\begin{vmatrix} x & y \\ \alpha_x & \alpha_y \end{vmatrix}}{\rho},$$

$$x_c = x - \frac{v^2 v_y}{\begin{vmatrix} v_x & v_y \\ \alpha_x & \alpha_y \end{vmatrix}}, \qquad y_c = y - \frac{v^2 v_x}{\begin{vmatrix} v_y & v_x \\ \alpha_y & \alpha_x \end{vmatrix}},$$

$$R = \frac{v^3}{\begin{vmatrix} v_x & v_y \\ \alpha_x & \alpha_y \end{vmatrix}}.$$

The reader will notice a certain symmetry running throughout this array which lends itself to easy memorization and use. The writer does not recall seeing these formulas set up in such manner in any text.

A NOTE ON AN ALGEBRAIC IDENTITY

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In his *Introduction to Mathematical Probability*, McGraw-Hill, 1937, p. 37, Professor J. V. Uspensky establishes indirectly by the principles of the theory of probability the following interesting identity for positive integers a and b:

$$(1) \qquad \frac{a+b}{a} = 1 + \frac{b}{a+b-1} + \frac{b(b-1)}{(a+b-1)(a+b-2)} + \cdots.$$

This identity can easily be shown to be a particular case of a much more general identity, from which an infinite number of other very interesting identities can be readily deduced.

Consider the obvious identity:

$$(2) \qquad \frac{b}{a} = \frac{b}{a+b+c}\left(1 + \frac{b+c}{a}\right).$$

Using (2) repeatedly, we deduce

$$\tag{3}\begin{aligned}\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{b}{a+b+c} + \frac{b(b+c)}{(a+b+c)(a+b+2c)} + \cdots \\ + \frac{b(b+c)\cdots(b+\overline{n-1}\,c)}{(a+b+c)(a+b+2c)\cdots(a+b+nc)} \\ + \frac{b(b+c)\cdots(b+nc)}{a(a+b+c)\cdots(a+b+nc)},\end{aligned}$$

from which (1) is obtained by letting $c=-1$. The series (1) terminates of itself.

If in using (2) repeatedly we let c take on successively the values $c_1, c_2-c_1, \cdots, c_n-c_{n-1}$, we obtain a more general result:

$$\tag{4}\begin{aligned}1 + \frac{b}{a} = 1 + \frac{b}{a+b+c_1} + \frac{b(b+c_1)}{(a+b+c_1)(a+b+c_2)} + \cdots \\ + \frac{b(b+c_1)\cdots(b+c_{n-1})}{(a+b+c_1)(a+b+c_2)\cdots(a+b+c_n)} \\ + \frac{b(b+c_1)\cdots(b+c_n)}{a(a+b+c_1)\cdots(a+b+c_n)}.\end{aligned}$$

(See J. A. Macdonald, "Note on the Summation of Finite Algebraic Series," *Mathematical Notes of the Edinburgh Mathematical Society*, no. 29, p. xiii, 1935, where a result equivalent to the above is established in a different way; and J. T. Hurt and L. R. Ford, "Polynomial expansions in the Borel region," *Proceedings of the Edinburgh Mathematical Society*, Series 2, vol. 5, part II, p. 82, 1937, where a similar procedure is used in expanding a rational fraction in a series.)

We note that by letting $c=0$ in (3) we deduce

$$\tag{5}\frac{b}{a} = \frac{b}{a+b} + \frac{b^2}{(a+b)^2} + \frac{b^3}{(a+b)^3} + \cdots ,$$

proving the well known fact that any rational fraction is the sum of an infinite geometric progression.

Again, by using two different sets of values for c's in (4) we obtain:

$$\tag{6}\begin{aligned}1 + \frac{b}{a+b+c_1} + \frac{b(b+c_1)}{(a+b+c_1)(a+b+c_2)} + \cdots \\ = 1 + \frac{b}{a+b+d_1} + \frac{b(b+d_1)}{(a+b+d_1)(a+b+d_2)} + \cdots .\end{aligned}$$

It may be added that the above results can be generalized still further by using repeatedly instead of (2) the identity:

$$\frac{b}{a}=\frac{b}{f(a,\,b)}\left[1+\frac{f(a,\,b)-a}{a}\right], \tag{7}$$

where either the same or a different function $f(a, b)$ is used in each successive application of the formula.

Finally, the identities (2) and (7) being valid for all values of a and b, and not only for positive integral values, the results deduced from them will be, with proper restrictions, of very great generality.

Since the series in (3) is readily expressed as a hypergeometric series of the type $F(\alpha, 1, \gamma, 1)$, it can be summed using the formula:

$$F(\alpha,\,1,\,\gamma,\,1)=\frac{\gamma-1}{\gamma-\alpha-1}, \tag{8}$$

and other identities can be deduced from (3) using the well known transformations of hypergeometric series. Also (1), (3), (4), (5), and (6) can be established by the use of more intricate methods of summation of algebraic series (Chrystal, *Algebra*, vol. II, chapter XXXI, and Knopp, *Infinite Series*, English translation, chapter VIII).

In this case the most elementary and the most direct algebraic methods seem preferable.

Note by the Editor. Professor Boldyreff has not considered the question of convergence in case his series are infinite. It might be interesting to examine the series

$$\begin{aligned}1&+\frac{b}{a+b+c_1}+\frac{b(b+c_1)}{(a+b+c_1)(a+b+c_2)}+\cdots\\&+\frac{b(b+c_1)\cdots(b+c_{n-1})}{(a+b+c_1)(a+b+c_2)\cdots(a+b+c_n)}+\cdots\end{aligned} \tag{4'}$$

for convergence. From (4) we see that the remainder in this expansion is

$$\begin{aligned}R_n&=\frac{b(b+c_1)\cdots(b+c_n)}{a(a+b+c_1)\cdots(a+b+c_n)}\\&=\frac{b}{a}\cdot\frac{b+c_1}{a+b+c_1}\cdots\frac{b+c_n}{a+b+c_n}.\end{aligned}$$

Hence

$$R_n^{-1}=\frac{a}{b}\left(1+\frac{a}{b+c_1}\right)\cdots\left(1+\frac{a}{b+c_n}\right).$$

Hence (4′) will converge if and only if $R_n\to 0$; that is, if and only if $R_n^{-1}\to\infty$. If there is an N such that $a/(b+c_n)\geqq 0$ for $n\geqq N$, a necessary and sufficient condition for this is the divergence of the series $\sum_1^\infty a/(b+c_n)=a\sum_1^\infty 1/(b+c_n)$. (Fine,

College Algebra, Ginn and Co., 1904, p. 566.) The series $\sum_1^\infty 1/(b+c_n)$ evidently diverges unless $|c_n|\to\infty$. If $|c_n|\to\infty$ there is an $M\geqq N$ such that $|c_n|>|b|$ for $n>M$. Then

$$2\left|\sum_M^\infty 1/c_n\right| = \left|\sum_M^\infty 1/(c_n/2)\right| > \left|\sum_M^\infty 1/(b+c_n)\right| > \left|\sum_M^\infty 1/2c_n\right| = \frac{1}{2}\left|\sum_M^\infty 1/c_n\right|.$$

Therefore $\sum_1^\infty 1/(b+c_n)$ diverges if and only if $\sum_M^\infty 1/c_n$ diverges.

As a special case we find that the infinite series

$$(3') \qquad 1 + \frac{b}{a+b+c} + \frac{b(b+c)}{(a+b+c)(a+b+2c)} + \cdots,$$

where $c_n = nc$, converges if a and c have the same sign.—R.J.W.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 513 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Gewöhnliche Differentialgleichungen. Dritte neubearbeitete Auflage. By Guido Hoheisel. Sammlung Göschen, Band 920, Berlin, Walter de Gruyter, 1938. 126 pages.

In addition to the usual treatment of elementary types of differential equations integrable by quadratures, this book treats such topics as the following: Existence and uniqueness theorems by the Picard method of successive approximations, integration of second order linear equations by series in the neighborhood of a regular singular point, and the Sturm-Liouville boundary value problem including a proof of the completeness of the systems of eigen functions and the absolute and uniform convergence of the corresponding series for functions having a derivative whose square is integrable over the interval in question.

The book is compact and reliable (save for a few obvious misprints) containing most of the standard material that an average student is likely to need. The proofs, though clear to the mature reader and in the simplest possible form, are perhaps too compact for the ready comprehension of a student meeting the subject for the first time. But accompanied by lectures and the companion volume (Sammlung Göschen, Band 1059) containing exercises, it should prove to be an excellent textbook.

D. C. LEWIS

Mathematical Adventures. By Fletcher Durell. Boston, Bruce Humphries, Inc., 1938. 157 pages. $2.00.

The first ten chapters of this book are reprints of articles published in *School Science and Mathematics* and *The Mathematics Teacher* from 1927 to 1932. The remaining two chapters were given as lectures before the Mathematical Club of Temple University and have not hitherto been published.

The goal of this book seems to be to present ideas which, if put into practice, would make mathematics a more rational, interesting, and vital subject to boys and girls in secondary schools; and in many respects this goal is reached.

Aside from this general aim, that which comes nearest to being a unifying idea of this rather loosely-knit book is what the author calls "cooperative mathematics." By this he means following the introduction of each new topic by a treatment of its relationships with other topics previously studied. This is well illustrated by the following exercise given on page 15: "An equilateral triangle, a square, a regular hexagon, and a circle all have the perimeter a. Obtain an expression for the area of each of these figures in the form of a^2 with a decimal or integral coefficient. Draw a bar graph to show the relative size of these areas. Find by what per cent each of the other areas exceeds the area of the triangle."

In the reviewer's opinion the chief faults in the book are due to too great a tendency to split a topic into its irreducible parts, each with its own label. This is carried to the greatest length in chapter five, where are listed fifteen different types of word problems classified according to the occurrence of the four fundamental operations; this is accompanied by the recommendation that they be taught separately (complete with their labels) and then combined. The chapter on the Fourth Dimension leaves one with the vague feeling that if he only knew more about this topic he could get out of his shirt without unbuttoning it and do many other useful things. In chapter six the author advances the idea that bright students who want merely a smattering of the subject would enter a section C along with poor students who could do no better; but, happily, he soon forsakes this notion.

On the other hand, there is enough worth while material in the book to make one forget its faults. In chapter seven the plea for a sensible geometry is an eloquent one; also, there may be found the excellent recommendation that "in beginning algebra" one should introduce as soon as possible "the equation as a useful instrument." The chapter on "Pleasure," stressing the importance of pleasure as a factor in learning and showing how it may be cultivated, is exceptionally fine. In chapter nine the thesis "in dealing with abstract ideas and laws . . . it is important to recur frequently and even systematically and periodically to the concrete meaning and uses of these general ideas" is very ably developed; in the reviewer's opinion, the failure to realize this principle is, more than any other one thing, at the root of the misunderstanding of mathematics among students today. Chapter eleven comes closest to the title of the book for it contains an excellent illustration of the intellectual and aesthetic enjoyment that can accompany a mathematical exploration.

Several of the chapters and parts of others are not only stimulating but a pleasure to read. This book should prove to be a very useful one.

B. W. JONES

Mathematical Snapshots. By H. Steinhaus. (Translated from Polish.) New York, G. E. Stechert and Company, 1938. 4+135 pages. $2.50.

Mathematical Snapshots is hardly to be classified under any of the usual categories. Indeed, the temptation is to make a new category—An Evening's Amusement—and to let this book be number one. The author uses pictures, figures, and models in addition to text to call attention to things of a mathematical or semi-mathematical character that are frequently overlooked or misunderstood. There is nothing like formal proof. The style is delightful, although there is an occasional lack of clarity, apparently introduced by translation. In fact, this book was more interesting to the reviewer than any book on so-called mathematical recreations that has ever come to his attention. Anyone, mathematician or what not, will enjoy turning its pages, looking at the pictures, and handling the models which are enclosed in a cover pocket, also reading the text. It is recommended to all, especially to those who are fond of puzzles and have a flair for mathematical recreations. The person who has never seriously studied mathematics, but who did well in high school mathematics and has a sneaking notion that he would have done well in more advanced work, will be delighted. Angle trisectors, circle squarers, and all such will find in it profitable channels for their abilities. Introduce them to it.

TOMLINSON FORT

Trigonometry. With Tables. By Howard K. Hughes and Glen T. Miller. New York, John Wiley and Sons, 1938. 8+189+79 pages. $2.00.

The first ninety-nine pages of this text are devoted to the solution of triangles and to those portions of the trigonometry which are required for this purpose. The trigonometric functions are defined first for the general angle, and the treatment of the reduction formulas is one which the student should have exceptionally little difficulty in mastering. The order of topics necessitates a geometric proof of the law of tangents, the usual proof being given later in the book. With the exception of one footnote, there is no mention of logarithms to bases other than ten.

Radian measure, trigonometric identities and equations, line values, graphs, and inverse functions occupy the next fifty-nine pages of the text. Noteworthy points in the section on identities are the emphasis on the restriction of the values of the unknown to those for which both members are defined, and the fact that the omission of the $\pm$ in the formula $\tan \frac{1}{2}x = \sin x/(1+\cos x)$ is justified.

There is no discussion of complex numbers, and the section on spherical trigonometry is very brief (19 pages). In the proof of the law of sines for spherical trigonometry, AE is written for CE, and sin a and sin b are interchanged. There is one misprint in §110.

The smooth, cream-colored paper is exceptionally pleasing and free from glare, and the print throughout the book and the tables is excellent.

ETHEL MOODY

Research and Statistical Methodology Books and Reviews of 1933–1938. Edited by Oscar Krisen Buros. New Brunswick, Rutgers University Press (1938). 6+100 pp. $1.25.

This book is a reprint of a section of the 1938 *Mental Measurements Yearbook of the School of Education of Rutgers University*. It consists of excerpts from reviews of "practically all the research and statistical methodology books published between January 1, 1933 and November 15, 1938 and written in the English language," to quote the words of the editor. Certainly a wide range of topics is treated; the discussion includes such dissimilar items as a manual on how to write a thesis and Wilk's Lectures on Statistical Inference. For each book the usual bibliographical information is given, including both American and English prices.

The reviews quoted are taken from a wide variety of journals; for instance, Sasuly's *Trend Analysis of Statistics* is discussed by means of excerpts from reviews which appeared respectively in *Agricultural Economics Literature*, the *American Economic Review*, the *Catholic Charities Review*, the *Journal of the American Statistical Association*, the *Journal of Political Economy*, and the *Journal of the Royal Statistical Society*. No complete list of all publications consulted is given, although the book is otherwise well indexed. Usually in the case of a mathematical book, among the excerpts which are given are quotations from the reviews which appeared in the *Mathematical Gazette* and in the *Bulletin of the American Mathematical Society* or in this MONTHLY. It might be mentioned in this connection that in the discussion of the Sasuly book, a profitable reference could have been made to the reviews by A. E. Waugh in this MONTHLY, vol. 42, 1935, pp. 505–507, and by K. W. Halbert in the *Bulletin of the American Mathematical Society*, vol. 41, 1935, pp. 607–610.

But on the whole, the excerpts selected for each book seem to give a fairly accurate picture of the book. Especial emphasis is placed on evaluative statements which direct attention to modern methods in the literature. A stated aim of the book is to furnish a guide to beginners and teachers of introductory courses in statistics which will enable them to select textbooks which are abreast of the modern developments. It seems to the reviewer that the book should be quite valuable for this purpose. However, by limiting the scope to reviews only of books published in the English language, all reference to the important

recent French and German treatises on probability and statistics is omitted; and this may tend to curtail the usefulness of the book from the standpoint of the more advanced student. Perhaps it will be found possible in later supplements to give excerpts of reviews written in English of important books published in other languages. In any case, the job as it now stands is well done, and seems to be well worth doing.

J. H. CURTISS

Statistical Methods, 2d edition. By G. W. Snedecor. Ames, The Collegiate Press, 1938. 13+378 pages. $3.75.

A review of the first edition by this writer appeared on pages 614–615 of the November 1938 MONTHLY. In the second edition, occasional statements throughout the book have been modified and clarified. In the way of additions, there is some excellent advice in the use of ratios and percentages, contained in what are now sections 1.15, 5.9, 6.18, 10.16 and the last part of 12.7. The "missing plot technique" is now given in sections 11.5 and 11.6. A number of other new topics are introduced also in Chapter 11, and three sections on multiple regression with more than three variables in Chapter 13.

W. E. DEMING

Geometrisieren im Bereiche wichtigster Kurvenformen. By Dr. Louis Locher-Ernst. Zürich und Leipzig, Orell Füssli, 1938. 62 pages.

The material of this book, according to its preface, was first presented in a series of lectures to an audience of laymen at the Goetheanum, Freie Hochschule für Geisteswissenschaft, Dornach. With trivial exceptions, each paragraph can be classified as expository or interpretative.

The expository paragraphs contain an extremely lucid synthetic account of the curves traced by a point moving in such a way that its distances (u, v) from two fixed points satisfy the equations: 1. $au+bv=\text{const}$, 2. $u^av^b=\text{const}$, 3. $v=ae^{bu}$. In each case, by a process which amounts to using a parameter measuring the time, the author has generated the curves in a dynamic and vividly intuitive way, with close attention to the heuristic motivation of each stage. No mathematical knowledge is assumed beyond the rudiments of plane geometry. The results are tabulated in an appendix.

The interpretative paragraphs will interest anyone concerned with the pedagogy or philosophy of mathematics, but will perhaps be received most warmly by those most sympathetic to the doctrines of anthroposophy and eurythmics, which, by liberal quotation, the author associates with Rudolf Steiner. The reviewer is obliged to profess quasi-complete ignorance of these doctrines, and will merely attempt briefly to suggest his own understanding of the general significance attributed to mathematics by the author.*

* Many other speculative questions are mentioned in the course of the discussion, along with a number of biographical incidents.

The book is motivated by the impulse "to augment the light of thought by the warmth of experience, [and] to illuminate the warmth of immediate experience by the clarity of thought." This interaction is felt to occur most conspicuously in experience which is "übersinnlich," here translated "over-sensory"; the author's preference of this adjective to a more neutral one seems instructive. The meaning of "übersinnlich" is indicated by examples: "The assertion 'the sum of the interior angles of a triangle is two right angles' gives vocal expression to an oversensory fact." The essence of mathematics, of geometry in particular, is held to consist in the treatment of sensory facts as symbols for comprehensive oversensory facts; experience of the former may be presented directly to anyone, whereas experience of the latter can neither be given to one unable or unwilling to grasp it nor confiscated from one who already enjoys it.* Herein is felt to consist the happiness most appropriate to the reflective soul. For the encouragement of human aspiration to oversensory experience, mathematics is to furnish evidence of the possibility of such experience, along with a [low] standard of the clarity and intensity it may be hoped to have. The technical material is presented and interpreted in the light of this doctrine.

The interpretative portion of the book thus treats problems which have not yet been formulated with the sharpness to which mathematicians have become habituated. Appraisals of its value—in general and in detail—will therefore differ greatly among mathematicians. The views suggested (rather than formally maintained) have received little attention from mathematicians, and their full explanation would require a statement much more elaborate than the author has been permitted to give. In its technical sections, the book appears to have the merit of offering the layman an immediate example of mathematical "beauty" more dynamic than that of elementary geometry.

F. A. Ficken

* This sentence and the two following are very free translations of passages selected for their aptness in illustrating the general tenor of the book, in its interpretative portions.

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N.J

CLUB REPORTS 1937–38

Mathematics Club, University of Cincinnati

The outstanding event of the calendar for the year was the testimonial dinner in honor of Professor Harris Hancock, who in June of 1937 retired from active teaching. As a permanent symbol of the esteem in which Professor Hancock is held, the Mathematics Club sponsored the painting of an oil portrait of him to be presented to the University.

During the year thirteen meetings were held, at which discussions centered around the following topics: Complex roots of equations, Hilbert's proof of the transcendence of e, A problem in dynamics, The cyclotomic equation, The harmonic oscillator in wave mechanics, Nomograms, Continuous geometry, A problem in physical chemistry, Geometric probability, A new type of sequence, Some parallelograms associated with a given parallelogram, Fermat's last theorem for the case $n=3$, Some properties of symmetric functions.

Mathematics Club, University of British Columbia

In line with its purpose of being designed to acquaint its members with branches of mathematics not dealt with in lectures and regular courses and also to stimulate individual investigation, the following papers were presented and discussed: An introduction to algebra, Mathematics underlying *Alice in Wonderland*, Generalized coördinates, History of mathematics, Magic squares, Calculus of variations.

Pi Mu Epsilon, Pennsylvania State College

Besides the various mathematics contests and social meetings sponsored by the chapter, the group centered their attention on the following problems presented for discussion: Mathematical solutions of an interesting problem, The history and evolution of π, The distribution of mass about an axis in a plane—about an axis in space, Elliptical harmonic motion, Newton versus Einstein in relativity, Mathematics in connection with vibrations, Tensor analysis.

Pi Mu Epsilon, University of Toledo

The club entertained at their February and April meetings two guest speakers. Professor W. L. Ayres of the University of Michigan gave the talk at the initiation banquet in February on the subject, The four color problem. Mrs. Robert Ayers of the Toledo Museum of Art at the later meeting talked on the subject, Dynamic symmetry showing relationships between mathematics and art.

Junior Mathematical Club, University of Chicago

Graduate students presented most of the talks at the regular meetings held throughout the year. The most interesting topics were: The geometric arches of E. H. Moore, Some identities of Cayley, Taylor and Laurent expansions, The Gamma function, Schwarz's lemma, Characteristic values for the third order linear homogeneous differential equation, Finite fields, Matrix approximation and the characteristic value problem, Historical trends in the Calculus of Variations, Convex and subharmonic functions, Moduls in quadratic fields and the composition of quadratic forms, Prime numbers. A joint meeting with the Physics Club was held, and informal discussions were encouraged at the customary afternoon tea preceding each meeting.

Pi Mu Epsilon, University of Georgia

The book *Mathematics for the Million* served as the basis for the programs for most of the meetings throughout the year. Mathematics—the mirror of civilization, Mathematics in prehistory, The history of algebra, The history of arithmetic, Statistics, were discussed by students, and miscellaneous subjects were chosen for the other meetings. Some of the most interesting were: Relations between mathematics and philosophy, Newton's *Principia* in a modern age, Mathematics in medicine, Relations between mathematics and English grammar. Dr. Messick of Emory University was the guest speaker at the initiation banquet in January. He talked on the subject "Descartes."

Harvard Mathematical Club, Harvard University

Many topics of interest to both undergraduate and graduate students of mathematics were discussed at the bi-weekly meetings of the club. Topics of lectures were: Harvard mathematicians and their works, Journals of mathematics, Postulational foundations of geometry, Famous mathematicians, Mathematical problems, Asymptotic expansion of functions defined by MacLaurin Series, Continued fractions, Binary notation in puzzles and games, Linear differential equations, Applications of group theory to differential equations, Dual numbers, Generalized Abelian groups, Critical points of harmonic functions.

Pi Mu Epsilon, University of Arkansas

The chapter held ten meetings during the school year. At most of these meetings talks were given both by student members of the organization and by various members of the faculty. Among the topics discussed were: Mathematics and engineering, Teaching of mathematics in secondary schools, Electrical computing machines, History of the calendar, Lives of mathematicians at the time of Napoleon, The fourth dimension, Place of mathematics in education. Two banquets were held during the year at which time new members were initiated.

Mathematics Club, Case School of Applied Science

Seven meetings were held throughout the school year, the first and last of which were combined business and social meetings held at the home of Dean T. M. Focke. At the special meetings, Dr. O. E. Brown and Dr. Max Morris of the department of mathematics spoke on the subjects, The application of determinants and projective transformations to nomography, and Complex elements in geometry, respectively. Other topics of meetings were: Geometric constructions, Probability, The theory of numbers, Continued fractions, Symmetric functions.

Pi Mu Epsilon, University of Kentucky

A gift of one hundred dollars to the departmental library was made by the organization as its regular yearly contribution to show its active interest in mathematics. A banquet and a picnic in honor of the initiates were the social activities of the year. At the eight regular academic meetings the topics on the program were: A theorem on direction fields, The configuration of double points of cubics of a pencil, Integral sets with all ideals principal, Polyadics, A theorem on ternary forms, Summation of divergent series, Irrational numbers, Topology.

Pi Mu Epsilon, University of Illinois

During the year there were twenty-five associate and forty-nine active members who met bi-monthly and heard the following papers: A brief history of Euclid's fifth postulate, A theorem concerning the greatest common divisor in Euclid's Algorithm, American mathematicians of today, An experimental formula concerning the expansion of binary forms, A criticism of Bell's *Men of Mathematics*, Mathematical economics, An historical survey of non-euclidean geometry.

Pi Mu Epsilon, Marquette University

The annual Frumveller examination was held in May. There were fifteen high schools from Milwaukee County represented, with seventy-eight contestants. The highest score was made by James Doran of Pulaski High School who was awarded the one-year scholarship to Marquette. At the six meetings during the year guest speakers presented a variety of interesting subjects: The power of reason, New applications of polarized light, The total eclipse of 1937.

Mathematics Club, Massachusetts State College

Of all the clubs listed with the department this one seems to be the most informal, having no officers, dues or outside speakers. Subjects for discussion are chosen most frequently by the students themselves, and they are allowed to work up the topic in their own way to develop their initiative and self-reliance. Six interesting meetings were held during the fall and winter terms, and reports were given upon a variety of subjects. Some of the most interesting follow: Certain higher plane curves, Solution of the cubic equation by Tartaglia, Hindu-Arabic notation, The mathematical exhibit at the Harvard Tercentenary, Dynamic symmetry, The Copernicus of antiquity, Alignment charts, The Richardson slide rule, Cartography, Theory of Numbers, Infinite series, Computation with large numbers.

Delta Nu Epsilon, Nebraska State Teachers College at Chadron

The faculty and student members of the club took turns in presenting papers at the bimonthly meetings. Mathematics in mechanics, Einstein's basic formulas, Prime numbers, Mathematics of chance in card games, Linkages, Topology, and Newton's method of extracting roots, were presented and gave rise to interesting discussions. Both the president and vice-president of the club won scholarships permitting them to work on advanced degrees at other schools for 1938–39.

Mathematics Club, Trinity College, Washington, D.C.

At most of the meetings held during the year, the subjects presented were discussed by outside speakers. Some of these were former members of the club. Their subjects were: Symmedian point of the triangle, Non-euclidean geometry, Teaching of algebra, Steiner's ellipse, Fermat's life and work, Mathematical puzzles, Inversion.

Kappa Mu Epsilon, Nebraska State Teachers College at Wayne

An introduction on the relations between mathematics and the various other departments of the college curriculum made up the program at one of the most interesting meetings of the year. Six members gave short talks on Art, Language, History, Chemistry, Biology, and Physics, and the part they play in mathematics. Further topics presented at other meetings during the year included: Famous mathematicians, The linear and the circular slide rule, Nomography, Use and operation of electric calculator, Geometry as a type of reasoning, and a debate: Resolved that the mathematical contributions of Archimedes have been more powerful in the world than those of Euclid.

Mathematics Club, St. Xavier College, Chicago

The lives and contributions of some of the great mathematicians and current views on the value and place of mathematics in the high school curriculum were the two general subjects for the papers presented during the year. The club helped in the mathematics exhibit sponsored by the Chicago Men's and Women's Mathematics Clubs at the Adler Planetarium.

Mathematics Club, Denison University

The program for the year follows: Intuitive geometry, Origin of mathematical notations used in trigonometry and calculus, Counting in higher mathematics, Parallel coördinates, Harmonic relations.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr. Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 383. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

The diameters from the vertices of the triangle ABC, in the circumscribed circle, cut the opposite sides in E, F and G respectively. L, M and N are the respective midpoints of AE, BF and CG. Show that triangle LMN is homologous to triangle ABC, and that the axis of homology is the orthic axis of the triangle.

E 384. *Proposed by V. Thébault, Le Mans, France.*

Determine the envelope generated by the polar line w of the fixed point W, with respect to the circle Q, which is of constant size, but rolls around a fixed circle C. What happens if circle C degenerates into a straight line, L?

E 385. *Proposed by C. W. Trigg, Los Angeles City College.*

What is the largest prime whose square contains no duplicate digits?

E 386. *Proposed by W. O. Pennell, Exeter, N. H.*

A certain irregular polygon has its n sides skewed in space. From each of its n vertices vectors are drawn to the mid-points of each of the $n-2$ non-adjacent sides. Show that the vector sum of these n^2-2n vectors is zero.

SOLUTIONS

E 346 [1938, 551]. *Proposed by Jack Lorell, Brooklyn, New York.*

Circles are constructed with the sides of a quadrilateral as diameters. Prove or disprove that at least one of the opposite pairs intersect.

Solution by C. W. Trigg, Los Angeles City College.

To disprove the proposition, it will suffice to cite one exception, an isosceles trapezoid with bases 2 and 26, equal legs of 13, and hence an altitude of 5. The distance between the midpoints of the legs is 13, whereas the sum of the radii of the circles on these legs as diameters is 13, so these opposite circles do not intersect. The distance between the midpoints of the bases, plus the radius of the circle on the smaller base, is 6, whereas the radius of the circle on the larger base is 13, so this circle wholly encloses the other. Therefore neither of the opposite pairs intersects, and the proposition is not true.

In their solutions, Paul Heinicke and E. P. Starke show that it is not possible for the two circles in each pair to be wholly external to each other, with no point in common.

E 347 [1938, 551]. *Proposed by A. V. Richardson, Bishop's College, Lennoxville, Que.*

The digits of a nine-place number are 1, 2, 3, 4, 5, 6, 7, 8 and 9, the order being determined by chance. Find the odds against the number being divisible by eleven.

Solution by H. D. Larsen, University of New Mexico.

In order that the number be divisible by 11, the sum of the 5 digits in the odd places must differ from the sum of the 4 digits in the even places by a multiple of 11. Since the nine digits add to 45, the two sets specified must add to 17 and 28 in one order or another.

A short calculation will show that there are 9 sets of 4 eligible digits adding to 17, and 2 sets adding to 28. Hence there are $11(4!)(5!)$ permutations of the 9 digits which yield a multiple of 11. The probability that a number is divisible by 11 is therefore $11(4!)(5!)/(9!)=11/126$, and the odds against it are 115 to 11.

Also received by W. E. Buker, Wm. Douglas, E. R. Heineman, Paul Heinicke, Herman Levy, E. P. Starke, C. W. Trigg and the proposer.

E 348 [1938, 551]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

In the triangle ABC, M is placed on AC, N on AB, and P on MN, in such a manner that $MC/AM=AN/NB=MP/PN=r$, a variable parameter. Show that P moves on a parabola, and determine its position with respect to the triangle.

Solution by J. A. Bullard, University of Vermont.

Let A be the point $(-a, 0)$, C the point $(a-h, -b)$, and B $(a+h, b)$. Then M falls at $[(a-ar-h)/(r+1), -b/(r+1)]$, and N falls at $[(ar-a+rh)/(r+1), br/(r+1)]$. The coördinates of P are consequently

$$x = a\left(\frac{r-1}{r+1}\right)^2 + h\left(\frac{r-1}{r+1}\right), \qquad y = b\left(\frac{r-1}{r+1}\right),$$

so that, eliminating $(r-1)/(r+1)$, we have $ay^2+hby-b^2x=0$.

This parabola is tangent to AC at C $(r=0)$, and is tangent to AB at B $(r=\infty)$. The point in which the parabola cuts the median through B is given by $r=\frac{1}{2}$, that in which it cuts the median through A by $r=1$, and that in which it cuts the median through C by $r=2$. By substituting the coördinates of the vertex in the above equations, we find that the vertex tangent is determined by $r=(2a-h)/(2a+h)$, that is, the ratio of the projection of AB on the median through A, to the projection of AC on that same median.

For further discussion of this parabola, see this MONTHLY (vol. 42, 1935, p. 606; vol. 44, 1937, p. 368).

Also solved by C. W. Bruce, Paul Heinicke, D. L. MacKay, Edward S. Smith (with twelve pages of ramifications), C. W. Trigg and W. I. Thompson.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3918. *Proposed by B. M. Stewart, University of Wisconsin.*

Given a block in which are fixed k pegs and a set of n washers, no two alike in size, and arranged on one peg so that no washer is above a smaller washer. What is the minimum number of moves in which the n washers can be placed on another peg, if the washers must be moved one at a time, subject always to the condition that no washer be placed above a smaller washer?

For $k=3$ this problem is called "The tower of Hanoï" in Ball's *Mathematical Recreations*, and the solution is given as 2^n-1.

3919. *Proposed by Richard Bellman, Brooklyn College.*

Prove that

$$\begin{vmatrix} \dfrac{x}{1-x} & 1 & 0 & 0 & 0\ \cdots & 0 \\ \dfrac{x^2}{1-x^2} & \dfrac{x}{1-x} & 2 & 0 & 0\ \cdots & 0 \\ \dfrac{x^3}{1-x^3} & \dfrac{x^2}{1-x^2} & \dfrac{x}{1-x} & 3 & 0\ \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots\ \cdots & \vdots \\ \dfrac{x^r}{1-x^r} & \dfrac{x^{r-1}}{1-x^{r-1}} & \cdot & \cdot & \cdot\ \cdots & \dfrac{x}{1-x} \end{vmatrix} = \frac{r!x^{r(r+1)/2}}{(1-x)(1-x^2)\cdots(1-x^r)}.$$

3920. *Proposed by F. A. Lewis, University of Alabama.*

Set

$$A_i = \sum_{j=1}^{4} a_{ij}x_j, \quad B_i = \sum_{j=1}^{4} b_{ij}x_j, \quad C_i = \sum_{j=1}^{4} c_{ij}x_j, \quad i = 1, 2, 3, 4,$$

where the x's represent independent variables and the determinant of the

coefficients of any four of the twelve linear forms does not vanish. Determine the number of distinct identities of the form

$$\alpha\prod_{i=1}^{4} A_i + \beta\prod_{i=1}^{4} B_i + \gamma\prod_{i=1}^{4} C_i \equiv 0$$

in the x's, where α, β, γ are arbitrary constants not all are zero, and each coefficient in the various linear forms is zero or an nth root of unity, where n is given in advance.

An example of such an identity is for $n=2$

$$(x_1^2 - x_2^2)(x_3^2 - x_4^2) - (x_1^2 - x_3^2)(x_2^2 - x_4^2) + (x_1^2 - x_4^2)(x_2^2 - x_3^2) \equiv 0.$$

An identity with the same coefficients in the various linear forms and $(\alpha, \beta, \gamma) = (k, -k, k)$ is not considered as distinct from the foregoing.

3921. *Proposed by V. Thébault, Le Mans, France.*

Let BCA_1A_2, CAB_1B_2, ABC_1C_2 be similar rectangles constructed upon the sides $BC=a$, $CA=b$, $AB=c$ of a triangle ABC of area S, the three rectangles being all directed interiorly or all exteriorly, and $CA_1/a=AB_1/b=BC_1/c=k$. Let A_h, B_h, C_h, be points on A_1A_2, B_1B_2, C_1C_2 such that $A_1A_h/A_1A_2=B_1B_h/B_1B_2 =C_1C_h/C_1C_2=\lambda$. The straight lines AB_h, BC_h, CA_h determine a triangle $\alpha\beta\gamma$ of area S' similar to ABC, and

$$S' = (k \cot V - \lambda)^2 S/(k^2 + \lambda^2),$$

where V is the angle of Brocard for ABC.

Problem 3850 [1937, 668] was incorrectly stated. Its last lines should be: Let A_3 be the symmetric of A_1 with respect to A_2, and similarly for B_3, C_3; then AB_3, BC_3, CA_3 meet in a point.

This follows from the present problem by taking $k=1$ for squares directed interiorly and $\lambda=2$.

SOLUTIONS

3826 [1937, 251]. *Proposed by W. Macray, Clark Academy, N. Y.*

Solve the system of partial differential equations,

$$U\frac{\partial W}{\partial x} + 2W\frac{\partial U}{\partial x} = 0, \qquad U\frac{\partial W}{\partial x} + 2W\frac{\partial V}{\partial y} = 0, \qquad \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = 0.$$

Solution by F. Underwood, University College, Nottingham, England.

From equations (1) and (2) in the problem we have (4) $U_x=V_y$. From (3) and (4) result (5) $V_{xy}=U_{xx}=U_{yy}$, which may be written $(D^2-D_1^2)U=0$, where $D=\partial/\partial x$, $D_1=\partial/\partial y$. The solution is (6) $U=f(x+y)+F(x-y)$. Then $V_x =f'(x+y)-F'(x-y)$, $V_y=f'(x+y)+F'(x-y)$. Hence (7) $V=f(x+y)-F(x-y) +C$. Writing (1) as $W_x/W=-2U_x/U$, we easily find (8) $W=\phi(y)/U^2$. The solution is given by (6), (7), (8) where f, F, ϕ are arbitrary functions and C is an arbitrary constant.

Solved also by D. G. Bourgin and Barbara Ellis.

Editorial Note. Bourgin added a discussion of the case where W depends on x alone, showing that the functional forms of f, F, and ϕ are almost completely prescribed. Setting $U=M(y)W^{-1/2}$ we have $U_{xx}=M(y)(W^{-1/2})_{xx}$, $U_{yy}=M''(y)(W^{-1/2})$. Hence $M''(y)/M(y)=(W^{-1/2})_{xx}/(W^{-1/2})$, and $M''(y)=k^2M(y)$.

Case A, where $k=0$. Then $M(y)=\alpha_1 y+\alpha_2$ and

$$W = (\beta_1 x + \beta_2)^{-2}, U = (\alpha_1 y + \alpha_2)(\beta_1 x + \beta_2).$$

We may also write

$$\begin{aligned} U = \tfrac{1}{4}\alpha_1\beta_1(x + y)^2 + \tfrac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1)(x + y) + \alpha_2\beta_2 \\ - \tfrac{1}{4}\alpha_1\beta_1(x - y)^2 - \tfrac{1}{2}(\alpha_1\beta_2 - \alpha_2\beta_1)(x - y). \end{aligned}$$

For V the signs of the terms in $(x-y)$ are changed and a constant C is added.

Case B, where $k\neq 0$. Here $M(y)=A_1 \cosh ky+A_2 \sinh ky$, $W(x)=(B_1 \cosh kx+B_2 \sinh kx)^{-2}$, and

$$\begin{aligned} U &= [A_1 \cosh ky + A_2 \sinh ky][B_1 \cosh kx + B_2 \sinh kx] \\ &= \tfrac{1}{2}(A_1B_1 + A_2B_2) \cosh k(x + y) + \tfrac{1}{2}(A_1B_2 + A_2B_1) \sinh k(x + y) \\ &\quad + \tfrac{1}{2}(A_1B_1 - A_2B_2) \cosh k(x - y) + \tfrac{1}{2}(A_1B_2 - A_2B_1) \sinh k(x - y). \end{aligned}$$

For V the signs of the terms with $(x-y)$ are changed and a constant C is added.

If W depends on y alone, it may be taken arbitrarily; and $U=hy+\alpha$, $V=hx+\beta$, where h, α, β are arbitrary constants.

3827 [1937, 252]. *Proposed by V. Thébault, Le Mans, France.*

With three consecutive integers taken from 0, 1, 2, $\cdots$, 9 form a number of five figures such that its square is formed from the ten given integers. The solution is unique.

Solution by C. W. Trigg, Los Angeles Junior College.

$1023456789<N^2<9876543210$, so $31992\leq N\leq 99387$. As N^2 contains the ten digits it is congruent to zero, modulo 9, so $N\equiv 0 \pmod 3$. Any triad of consecutive digits is divisible by 3; so N must be a permutation of three consecutive digits and their member which is a multiple of 3 repeated twice, or of the three digits and the non-multiples of 3 repeated once each. From a table of squares of the numbers from 1 to 10,000 can be read the first three (in general) and the last four digits of N^2 for the 300 eligible values of N. Duplicate digits are found in all but twenty of these, which when squared reveal the unique value $(55446)^2=3074258916$.

Solved also by W. E. Buker, E. P. Starke, and the proposer.

3828. [1937, 252]. *Proposed by V. Thébault, Le Mans, France.*

(a) The perpendiculars from a point P to the straight lines AQ, BQ, CQ, which join the vertices of a triangle to an arbitrary point Q, cut the sides of the triangle $A_1B_1C_1$, determined by the perpendiculars to the lines PA, PB, PC,

drawn from the inverse points of A, B, C in an inversion (P, k), in three points of a straight line Δ perpendicular to PQ.

(b) If the point P remains fixed and Q describes a given straight line d, the straight line Δ passes through a fixed point.

Solution by Otto J. Ramler, The Catholic University of America.

(a) Draw the circles having PA, PB, PC as diameters. Let QA, QB, QC meet these circles in A', B', C' respectively, Then since PA', PB', PC', are perpendicular to QA, QB, QC respectively it readily follows that P, Q, A', B', C' are on a circle having PQ for a diameter. Now subject the four circles (AP), (BP), (CP), and (QP) to an inversion (P, k). The circles (AP), (BP), (CP), invert into the straight lines B_1C_1, C_1A_1, A_1B_1, respectively determined by the perpendiculars to the lines PA, PB, PC, drawn from the inverse points of A, B, C in the inversion (P, k). The points A', B', C' invert into the intersections of PA', PB', PC', with B_1C_1, C_1A_1, A_1B_1 respectively, and since A', B', C', P lie on a circle through the center of inversion it follows that their inverses lie on a straight line Δ. Since under the transformation of inversion angles are preserved, it follows that since PQ is a diameter of the circle $(A'B'C'PQ)$ and inverts into itself, the line Δ must be perpendicular to PQ.

(b) If Q describes a straight line d, the circles having PQ as diameter all pass through P and F, the foot of the perpendicular from P upon d. Hence, the corresponding lines Δ all pass through a fixed point, the inverse of F as to the circle of inversion.

Solved also by W. C. Janes, L. M. Kelly, and the proposer.

Editorial Note. Janes's solution used rectangular coördinates, while the proposer and Kelly used the geometry of inversion. It is simpler to use polar properties. Thus ABC and $A_1B_1C_1$ are polar reciprocal triangles with respect to the circle (P) with the center P. Let the perpendicular from P to QA cut B_1C_1 in $\overline{A}$. Then the polar of $\overline{A}$ goes through A and it is perpendicular to $\overline{A}P$. Hence the polar of $\overline{A}$ is QA; and Δ, the polar of Q, must be the perpendicular through $\overline{A}$ to PQ. If $\overline{B}$ and $\overline{C}$ are defined in the same way as $\overline{A}$, these three points lie on Δ. If Q moves on a straight line d, then Δ, the polar of Q, must pass through D, the pole of d. This concludes the proof.

3830. [1937, 332]. *Proposed by Otto Dunkel, Washington University.*

In n dimensional euclidean space, $n \geqq 2$, to the simplex S there corresponds a simplex S' such that the perpendiculars from each vertex A_i of S to the face of S' opposite A_i', $1 \leqq i \leqq n+1$, meet in a point P. Show that the perpendiculars from each vertex A_i' of S' to the face of S opposite A_i meet in a point Q; that is, S and S' are orthologic.

Solution by the Proposer.

It is assumed that neither S nor S' is degenerate, and this assumption will be used in the proof. Let the vectors from any chosen origin to the vertices of S and S' be $\mathbf{a}_i$ and $\mathbf{a}_i'$; and let the vector of P be $\mathbf{y}$. Then

$$(1) \qquad (\mathbf{y} - \mathbf{a}_i)\cdot(\mathbf{a}_j' - \mathbf{a}_k') = 0, \qquad i, j, k \text{ distinct.}$$

There are $(n+1)n(n-1)/2$ such equations, which must be consistent. We suppose first that n is greater than two so that there are more than three distinct subscripts. Replacing i in (1) by h, we obtain from the two equations

$$(2) \qquad (\mathbf{a}_h - \mathbf{a}_i)\cdot(\mathbf{a}_j' - \mathbf{a}_k') = 0, \qquad h, i, j, k \text{ distinct,}$$

as a necessary condition for the existence of the vector $\mathbf{y}$. There are $(n+1)n(n-1)(n-2)/4$ such equations. We now show that the equations (2) suffice to determine $\mathbf{y}$. Consider the equations.

$$(3) \qquad \begin{aligned} (\mathbf{y} - \mathbf{a}_i)\cdot(\mathbf{a}_1' - \mathbf{a}_{i+1}') &= 0, \qquad i = 2, 3, \cdots, n, \\ (\mathbf{y} - \mathbf{a}_{n+1})\cdot(\mathbf{a}_1' - \mathbf{a}_2') &= 0. \end{aligned}$$

Since the simplex S' is non-degenerate, there exists a unique solution of the system (3) for the vector $\mathbf{y}$. Any one equation of (3) with a suitable one of (2) gives $(\mathbf{y}-\mathbf{a}_h)\cdot(\mathbf{a}_1' - \mathbf{a}_{i+1}')=0$, $1, h, i+1$ distinct. Two of the latter give $(\mathbf{y}-\mathbf{a}_h)\cdot(\mathbf{a}_j'-\mathbf{a}_k')=0$, $1\neq h, j, k$ distinct. Then from $(\mathbf{a}_h-\mathbf{a}_1)\cdot(\mathbf{a}_j' - \mathbf{a}_k')=0$, we have $(\mathbf{y}-\mathbf{a}_1)\cdot(\mathbf{a}_j' - \mathbf{a}_k')=0$; and finally these results give $(\mathbf{y}-\mathbf{a}_h)\cdot(\mathbf{a}_j' - \mathbf{a}_k')=0$, for h, j, k distinct. Hence the $\mathbf{y}$ determined by (3) satisfies the system (1). But this shows also that (2) is a necessary and sufficient condition for the existence of Q, and Q is uniquely determined by a system similar to (1).

For $n=2$ there are three equations $(\mathbf{y}-\mathbf{a}_i)\cdot(\mathbf{a}_j' - \mathbf{a}_k')=0$, $i, j, k=$a cyclic permutation of 1, 2, 3. As before there exists a unique solution for $\mathbf{y}$ of a selected pair of these equations; and a necessary and sufficient condition for consistency of the three equations is $\sum \mathbf{a}_i\cdot(\mathbf{a}_j' - \mathbf{a}_k')=0$. This is equivalent to $\sum \mathbf{a}_i' \cdot(\mathbf{a}_j-\mathbf{a}_k)=0$. Hence the two triangles are orthologic, and the proof is now complete for $n\geqq 2$.

If in the first case where $n>2$, we take $\mathbf{a}_i=\mathbf{a}_i'$ so that $S\equiv S'$, the condition (2) means that S is orthocentric with $\mathbf{y}$ as the vector of the orthocenter. Since every triangle is orthocentric, we must have a different form of condition which becomes an identity when the two triangles coincide; and such is the case in the condition above for triangles.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

A meeting of mathematicians of Pacific Northwest institutions was held at the University of Washington on April 1 and 2, 1939. The following mathematicians presented papers: Professor J. P. Ballantine, Dr. Harold Chatland, Harry Goheen, Professor F. L. Griffin, Dr. O. G. Harrold, Dr. P. G. Hoel, Professor Ralph Hull, Professor M. S. Knebelman, Dr. A. H. Taub, and R. L. Wertz. It was decided to have an annual meeting, to have the next meeting at Reed College in Portland, Oregon, and to consider at the next meeting the question of affiliation with one of the national mathematical groups.

The members of the Mathematical Association may recall that in 1932 a translation of the First Carus Monograph, *Calculus of Variations* by Professor G. A. Bliss, was published by B. G. Teubner of Berlin through special arrangement between the publisher and the Mathematical Association. The price of this translation by Dr. F. Schwank of Frankfurt is 7 marks, but through a special discount it is available to members of the Association for 3.70 marks. This edition incorporates a number of improvements made by Professor Bliss after the appearance of the Monograph in 1925.

A new publication outlet for manuscripts dealing with science and technology has been provided at Iowa State College, Ames, Iowa, by the recent organization of the Iowa State College Press, whose major purpose is "to serve learning, and particularly learning in fields of science and technology." The new press will consider for publication manuscripts from any source. The manufacture and sale of Iowa State College Press publications will be conducted by the Collegiate Press, Inc., Ames, Iowa.

Professor A. E. Landry of the Catholic University of America represented the Mathematical Association of America at the one hundred fiftieth anniversary of the founding of Georgetown University, May 28 to June 3, 1939.

Professor R. H. Reece of the New Mexico School of Mines represented the Mathematical Association at the fiftieth anniversary of the founding of the University of New Mexico, June 4–5, 1939.

Associate Professor L. M. Graves of the University of Chicago has been promoted to a full professorship.

Assistant Professor R. L. O'Quinn of Louisiana State University has been promoted to an associate professorship.

The following appointments to instructorships for the year 1939–40 are announced:

Bradley Polytechnic Institute: Ralph Johanson.
University of Chicago: Dr. O. F. G. Schilling.

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